This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

GENERAL MATHEMATICS

BOOK TWO

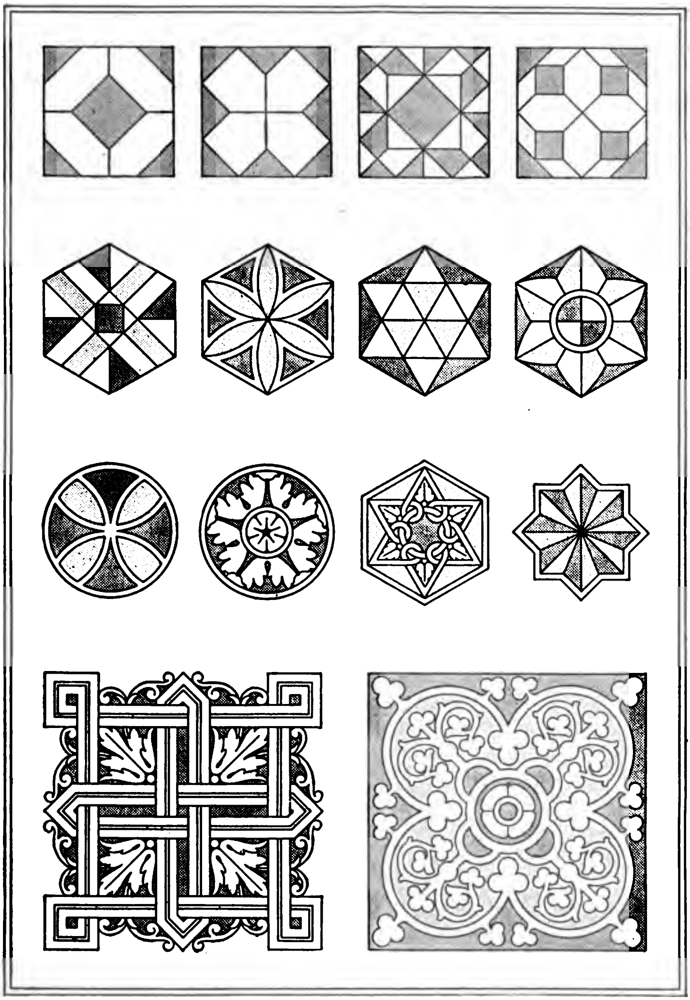
GINN AND COMPANY

Educ T 109.22.740

HARVARD COLLEGE
LIBRARY

THE GIFT OF
ALBERT E. WINSHIP, LITT.D., LL.D.
EDITOR OF THE NEW ENGLAND JOURNAL
OF EDUCATION

3 2044 096 989 926



GEOMETRIC FIGURES COMBINED INTO ATTRACTIVE DESIGNS

GENERAL MATHEMATICS

BOOK TWO

BY

WILLIAM DAVID REEVE
UNIVERSITY HIGH SCHOOL, UNIVERSITY OF MINNESOTA

GINN AND COMPANY
BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

Educ T 109.22.740
~~Math 359.22~~

HARVARD COLLEGE LIBRARY
GIFT OF
ALBERT EDWARD WINSHIP
December 19, 1922

COPYRIGHT, 1922, BY
WILLIAM DAVID REEVE

ALL RIGHTS RESERVED

622.8

The Athenæum Press
GINN AND COMPANY · PRO-
PRIETORS · BOSTON · U.S.A.

PREFACE

This book is a companion volume to Schorling and Reeve's "General Mathematics," Book One. The material included has been given a careful trial by the author in the classroom and constitutes what has proved to be a thoroughly usable course in general mathematics for the second year, or tenth grade, of the high school. There is, however, no reason why this book should not be used in second-year classes which have had instruction in algebra only during the first year.

While the emphasis in the first book was placed largely upon algebra, the emphasis here is placed principally upon geometry. At the same time, however, the earlier work in arithmetic, algebra, and trigonometry is kept before the student by means of reviews and applications, and enough new material is introduced to give to the student that knowledge of the fundamentals of these subjects which he should have at the end of his second year's work.

The fundamental laws and operations of arithmetic are kept before the student by their uses in connection with the development of formulas, new topics in algebra, and applied problems.

The algebra of the first year is reviewed and extended by the application of algebraic notation to geometric situations, by the introduction of new topics such as the solution of quadratic equations by formula, and by the work on fractions and trigonometric identities.

Numerical trigonometry, which was begun in the first book, is here reviewed and extended. The student in the high school has the right to learn how to use this most efficient tool in solving problems, and the trigonometry included in this course is as simple as the algebra and geometry of the first two years.

The course in plane geometry is concluded in this book. After the first chapter the geometry work is more formal than was the case in the first book, and thus the informal experimental type of proof by measurement is replaced gradually by the demonstrative type which is independent of measurement. Some work on optical illusions and geometric fallacies is given to put the student on his guard against making careless deductions.

At present most children leave school without an adequate conception of the spatial relations about them, and for this reason the fundamental ideas of lines and planes in space are included in this course. Plenty of time has been saved for this work by the unification of the various subjects and by the elimination of many topics that heretofore have been included in the different branches of mathematics when taught separately but which are unnecessary and too difficult in a course in general mathematics. Moreover, many of the propositions of solid geometry are analogous to those of plane geometry and can be easily taught in connection with them. In addition, a simple introduction to the most elementary type of analytic geometry is given in order that the student may gain some familiarity with the idea of the correspondence existing between an equation and a line of the plane, and may recognize the type of conic represented by an equation of the second degree.

The aim of the general mathematics course is to give the student *not so much mathematics* as is generally given in the traditional courses but *more about mathematics.* In other words, the student should know mathematics as a tool to be used rather than as a group of related subjects, such as algebra, geometry, trigonometry, and so on. With such an aim it has been possible to arrange this course so as to make unnecessary the long tedious reviews that are so often given and to save the time thus spent for the introduction of new and interesting material. Throughout the book a special effort is made to impress the student with the various uses that are made of mathematical laws and principles in everyday life, and many exercises which show the practical uses of geometry are included.

The author feels that the following points are features of this book that merit consideration:

1. A simple, informal, and inductive approach to the year's work is afforded by means of an introduction (Chapter I) which most children can understand and enjoy. Then Chapter II begins to teach the more rigorous methods of demonstrative geometry without making the student feel that he is an incompetent in a strange world, as is so often the case.

2. Definitions are introduced only when the need for them is apparent and, as far as possible, are made the outgrowth of the work rather than the basis of it.

3. In order to give the student skill in reading, understanding, and attacking propositions a very thorough discussion and illustration of the various methods of proof used in mathematics is given. Many points are explained which the student must know if he is to master the art of logical procedure in his work, and especially if he is to enjoy doing it. Many students would do much better work and would enjoy the precision of geometry if they had an intelligent idea of how to proceed. It is the author's intention to give the student an idea of systematic planning in his work and to imbue him with the idea that mere stumbling along and trusting to luck or memory will not avail.

4. In order to insure an efficient and systematic planning of the work, every proof of a theorem that is a complete proof is in reality a *model.* The statements and reasons for the statements are given in full instead of referring the student to some previous axiom, definition, or proposition. Frequently students do not look up these references, and thus lose much that they

will gain by the method used in this book. Many theorems have been left for the student to prove, hints or suggestions as to the method to be used being given where desirable.

5. The form of the proof, although not at all a new idea, is a feature. The reasons in the proof are placed in a second column at the right of the statements. This method, which will be used by the student in writing a proof, makes the reading of papers a much easier task than any other yet devised.

6. A very careful psychological procedure in the application to problems of the three so-called *fundamental theorems of plane geometry* (Arts. 108–124) is developed. The work starts with exercises where the student needs to make only one step to obtain the proof, then two steps, three steps, and so on.

7. Much traditional material that is now considered useless or of little importance either in mathematics or in any of the allied fields is omitted. The theory of limits and the proofs of all incommensurable cases are omitted in this course as being too difficult. In accord with the best modern thought along this line, the commensurable cases are proved and the incommensurable cases are discussed and assumed without proof.

The author is indebted to the editorial staff of Ginn and Company for many valuable suggestions and much help, without which the book could not appear at this time. He is especially indebted to Mr. E. L. Morss. While especial care has been taken to have the text free from errors, the author will be grateful to any teacher who may wish to give suggestions for improving the book in any way.

THE AUTHOR

MINNEAPOLIS, MINNESOTA

CONTENTS

CHAPTER PAGE

I. INTRODUCTION 1

What we study in geometry 1

Measurement of angles; construction of angles; the protractor 14

Parallel lines 23

Sum of the angles of a triangle 36

Classes of triangles 41

Quadrilaterals 42

II. CONGRUENCE OF RECTILINEAR FIGURES 48

Necessity of establishing truths by reasoning 48

Optical illusions 49

Demonstrative geometry 51

Superposition; congruence defined 52

Symmetry 56

Fundamental tests of congruence 59

Construction lines 67

Method of proving propositions 70

Principle of Duality 79

Converse theorems 86

Synthetic and analytic methods 92

Geometrical fallacies 96

III. POLYGONS 99

Inductive method 99

Classification of quadrilaterals 105

Summary of properties of a parallelogram 110

Law of parallelogram of forces 114

Prismatic surface; prismatic space; prism 119

Inequalities 128

Opposite of a proposition 136

CHAPTER PAGE

IV. LOCI OF POINTS 138
Method of proving a locus 141
Concurrent lines 145
Intersection of loci 154
Terms used in graphical representation 157
Parabola 159
Solution of quadratic equations by formula 160
The circle as the graph of a quadratic equation . . . 163
Simultaneous equations 165
Ellipse 167
Hyperbola 171

V. LINES AND PLANES IN SPACE. DIHEDRAL ANGLES . . 177
Skew lines; skew polygons 180
Direct and indirect methods of proof 181
Kinds of dihedral angles 201
Perpendicular planes 202
Projections 208

VI. STRAIGHT LINES AND CIRCLES 212
Fundamental definitions 212
Tangent circles 222
Principle of Continuity illustrated 228
Geometric constructions 240
Necessary and sufficient conditions for determining a triangle 242
Use of loci in geometric constructions 245

VII. RATIO; PROPORTION; VARIATION; SIMILAR POLYGONS . 262
Commensurable and incommensurable magnitudes . . 264
Means and extremes 265
Mean proportional 266
Direct variation 270
Inverse variation 271
Harmonic division of a line segment 281
Similar figures 286

CHAPTER PAGE
Diagonal scale 298
Pantograph 299

VIII. TRIGONOMETRY 308
Scale drawing 308
Sine of an angle 311
Cosine of an angle 312
Tangent of an angle 312
Table of trigonometric functions 315
Functions of 30°, 45°, and 60° 317
Solving right triangles 320
Applied problems 324

IX. LAWS OF FRACTIONS; FRACTIONAL EQUATIONS . . 328
General law for the addition and subtraction of fractions 334
Applying the laws of fractions to formulas 338
Law for multiplying one fraction by another 340
Law for dividing one fraction by another 343
Complex fractions 344
Trigonometric identities 348
Trigonometric equations 352

X. AREAS 354
Area of a rectangle 355
Area of a parallelogram 357
Area of a triangle 358
Area of a trapezoid 358
Area of polygons 359
Trapezoidal rule 361
General angle 362
Trigonometric functions of an obtuse angle 363
Various methods of proving the Pythagorean theorem . 366
Dependence illustrated geometrically 368
Law of cosines 371
Distance formula 376
Midpoint formulas 377
Law of sines 383

CHAPTER PAGE
XI. INSCRIBED AND CIRCUMSCRIBED REGULAR POLYGONS . 401
  Dividing a circle into equal parts 403
  Extreme and mean ratio 410
  Circumference of a circle 422
  Finding the area of a circle 429
  Radian measure 430

TABLE OF ROOTS AND POWERS 439

INDEX 441

GENERAL MATHEMATICS

BOOK TWO

CHAPTER I

INTRODUCTION

1. What we study in geometry. In arithmetic we work primarily with numbers (computation). For example, we say that the area of a triangle whose base is 6 in. and whose height is 4 in. is $\frac{6 \times 4}{2}$ sq. in.

In algebra we are not primarily interested in the area of any particular triangle, as in the preceding case, but we are interested in the general rule for finding the area of a triangle; namely, $A=\frac{bh}{2}$, where A is the area in square units and b and h represent the number of units in the base and the height respectively.

In studying geometry we deal primarily with four fundamental concepts; namely, points, lines, surfaces, and solids. We call these fundamental concepts *geometric magnitudes*. We may also deal with combinations of these concepts or with these concepts in motion.

A moving point is said to *generate* (describe) a line. In like manner, if a line moves in any direction, except along itself, it is said to generate a surface. And if a surface moves in any direction, except along itself, it is said to generate a solid.

2. Length. Length is the important characteristic of a line. In an exact sense a line has length only, not width nor thickness. Thus, the edge of a table has length only; the thickness and width of a crayon line are neglected; the wide chalk marks on a tennis court are not boundary lines, but are made wide to help us see the real boundary lines, which are the outside edges of the chalk marks.

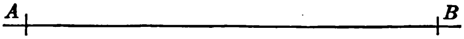


FIG. 1. A LINE SEGMENT

The part of a line whose length we wish to determine is a *line segment* or, briefly, a *segment*, as AB in Fig. 1.

A line segment has a definite beginning point and a definite ending point. The word "point" is used to mean merely position, not length, breadth, nor thickness. The position of a point is shown by a short cross line; that is, *a point is determined by two intersecting lines:*

NOTE. The pupil should now provide himself with a ruler one edge of which is graduated to inches and fractional parts of an inch and the other to units of the metric scale. He should also obtain a pair of compasses and some squared paper ruled to the metric scale.

3. Collinear points. Points which lie in the same straight line are said to be *collinear.*

4. Ray; origin. If a point divides a straight line into two parts, each part is called a *ray.* The rays are said to have only one end and to extend indefinitely in opposite directions from that end, as shown in Fig. 2. The point O from which the rays extend is called the *origin.*

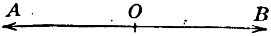


FIG. 2

5. Broken line. A *broken line* is a line which is not straight, but one which is made up of two or more different straight lines (see Fig. 3, 1).

6. Curved line. A *curved line* is a line no part of which is straight (see Fig. 3, II).

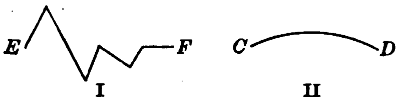


FIG. 3

7. Properties of straight lines; postulates. The following properties of straight lines are accepted without proof. Such assumptions when they relate to geometric magnitudes are called *postulates.*

1. *Any number of straight lines can be drawn through a point.*
2. *Two points determine one and only one straight line.*
3. *Two straight lines cannot intersect in more than one point.*
4. *The length of the line segment connecting two points is the shortest distance between them.*

EXERCISES

1. Explain what is meant by postulate 2 above.

2. Why must postulate 3 above be accepted as evident?

3. Why does a gun have two sights on it?

4. How may three posts be set in a straight line by means of postulate 2 above?

5. How many straight lines are determined (fixed) by three points that are not collinear? Illustrate.

6. How many straight lines are determined by four points that are not collinear? Illustrate.

8. Plane. The student is so familiar with the notion of a plane surface or plane that it is difficult to find more common words or better-known concepts by means of which we may define it. We are all familiar with the well-known test for planeness of a surface; namely, the application of a *straightedge* to the surface in different directions. This test depends upon the obvious fact that if any two points in the surface are taken, the straight line joining them lies wholly in that surface. Thus, in Fig. 4, if A and B are two points of the plane π,[1] the entire straight line AB lies in the plane π. Hence, if a straightedge touches a surface at all points in every position, the surface is called a *plane surface* or a *plane*. A flat surface such as the floor, the ceiling, or the top of a table is a good illustration of a plane surface or plane. Use your ruler and test for planeness the surface of your desk, a ball, and other objects.

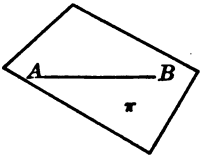


FIG. 4. A PLANE

9. *Postulate.* It is evident from the preceding discussion that *if a line has two of its points in a plane, it lies wholly in that plane.*

EXERCISES

1. Point out examples of plane surfaces that are in the classroom; of surfaces that are not plane.

2. How may a surface that is slightly convex be tested?

3. How does a carpenter test a floor for planeness?

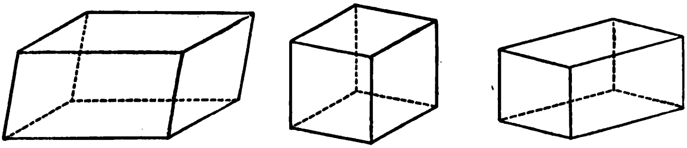
4. Why does a mason use a trowel with long, straight edges when he is testing a wall for planeness?

5. Are there any two points on the surface of a ball such that a straight line connecting them lies on the surface of the ball?

10. Solid. The drawings in Fig. 5 represent geometric solids. A *solid* is commonly thought of as an object that occupies a portion of space, although the space it occupies is in reality the solid, which is separated from the surrounding space by its surface. In geometry we study only the form and size of a solid since we are not interested in

---

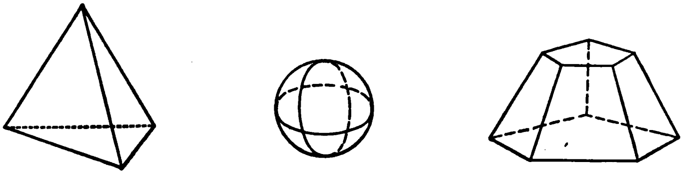
[1] From now on in this text we shall use a single Greek letter to designate a plane. Two letters are often used, but there is no lack of clearness if we use only one and that a Greek letter.



Oblique Parallelepiped

Cube

Rectangular Parallelepiped



Triangular Pyramid

Sphere

Frustum of a Pyramid

FIG. 5. SOLIDS

its color, its weight, in the matter composing it, and so on. In fact, a geometric solid is purely a mental concept.

11. Geometric figure. Any geometric point, line, surface, solid, or combination of these concepts is called a *geometric figure.*

12. Geometry; plane and solid geometry; coplanar. Geometry is the science which deals with geometric figures. In *plane geometry,* or the geometry of two dimensions, we deal with figures whose parts lie in the same plane. When the parts of a geometric figure lie in the same plane they are said to be *coplanar. Solid geometry,* or the geometry of three dimensions, treats of figures whose parts are not all in the same plane. Solid geometry is sometimes called the *geometry of space.*

EXERCISES

1. Point out figures the parts of which are coplanar that you find in the classroom.

2. Point out figures the parts of which are not coplanar that you find in the classroom.

3. How can a point move so as not to generate a straight line?

4. How can a straight line move so as not to generate a surface?

5. Can a curved line or a broken line move in space so as not to generate a surface?

6. How can a plane move in space so as not to generate a solid?

13. Rectilinear figure. A plane figure, all of the lines of which are straight lines, is called a *rectilinear figure*. Thus, the triangle ABC (Fig. 6) is a well-known rectilinear figure, formed by joining the three points A, B, and C by line segments. The three points A, B, and C are called *vertices*.

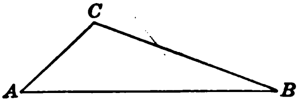


Fig. 6

14. Curvilinear figure. A plane figure formed by a curved line is called a *curvilinear figure*. Thus, the circle (Fig. 7) is a curvilinear figure.

15. Boundaries of a geometric figure. The points, lines, or surfaces which inclose a geometric figure are called its *boundaries*. Thus, the boundaries of a line segment are points, those of a surface are lines, and those of a solid are surfaces.

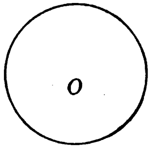


Fig. 7

16. Polygon. A rectilinear figure, as $ABCDE$ (see Fig. 8), formed by connecting points, as A, B, C, D, and E, by line segments, is a *polygon*. The Greek phrase from which the word "polygon" is derived means *many-cornered*. A polygon having 3, 4, 5, 6, 8, 10, · · ·, n sides is called a *triangle*, a *quadrilateral*, a *pentagon*, a *hexagon*, an *octagon*, a *decagon*, · · ·, an *n-gon* respectively. The sum of the sides of a polygon is its *perimeter*; that is, the perimeter is the length of the boundary. When all the sides of a polygon are equal, the polygon is said to be *equilateral*. When all the angles of a polygon are equal, it is said to be *equiangular*. When a polygon is both equilateral and equiangular, it is said to be *regular*.

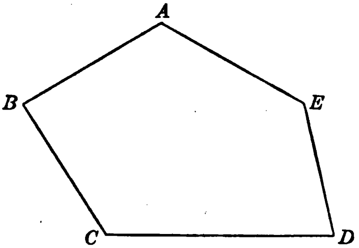


Fig. 8. A Polygon

17. Compasses. A pair of compasses (Fig. 9) is an instrument that may be used in measuring line segments or in drawing circles. A good substitute for the compasses, if they are not available, is an ordinary string with a loop tied in one end into which a pencil or crayon may be inserted.

Note. It is understood that only two instruments are to be used in geometry; namely, the ruler and the compasses.

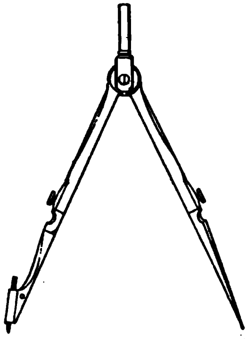


Fig. 9. A Pair of Compasses

18. Measuring a line segment with the compasses. In measuring with the compasses, the screw in the head should be adjusted so that the legs will remain in any position in which they are set. To measure the line segment AB in

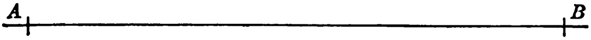


Fig. 10

Fig. 10 place the sharp points of the compasses on A and B, then place the points on the marks of the ruler and count the number of inches or centimeters between them.

19. Squared paper. Squared paper is another important device which is often useful in measuring line segments.

Squared paper is ruled either to inches and fractions of an inch or to the units of the metric scale. A part of a sheet is shown in Fig. 11. The method of measuring with squared paper is practically the same as that of measuring with the compasses and ruler. Thus, to measure the line segment AB in Fig. 11, place the sharp points of the compasses on A and B. Then place the sharp points on one of the heavy lines, as at E and D. Each side of a large square being 1 cm., count the number of centimeters in ED and estimate the remainder to tenths of a centimeter. In this way we find that the segment AB is 2.9 cm. in length.

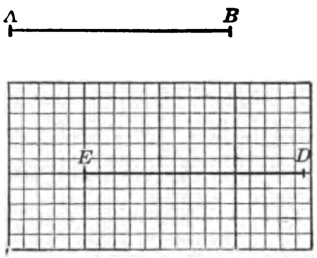


FIG. 11. HOW A LINE SEGMENT MAY BE MEASURED BY THE USE OF SQUARED PAPER

20. Equal line segments. When the end points of one segment, as a in Fig. 12, coincide with (exactly fit upon) the ends of another segment, as b, the segments a and b are said to be *equal*. This fact may be expressed by the equation $a = b$.

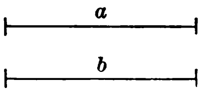


FIG. 12. EQUAL LINE SEGMENTS

21. Unequal segments; inequality. If the end points of two segments, as a and b, cannot possibly be made to coincide, the segments are said to be *unequal*. This is written $a \neq b$ (read "a is not equal to b"). In Fig. 13 segment a

is less than segment b (written $a < b$), and segment c is greater than segment b (written $c > b$). The statements $a \neq b$, $a < b$, and $c > b$ are called *inequalities*.

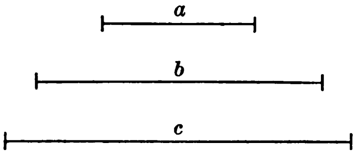


FIG. 13. UNEQUAL LINE SEGMENTS

In the preceding paragraph we need to remember that the letters a, b, and c stand for the length of the segments. They represent numbers, which are determined by measuring the segments.

22. Angle. If a straight line, as OX in any of the drawings of Fig. 14, rotates in a plane about a fixed point, as O, in the direction indicated by the arrowheads until it

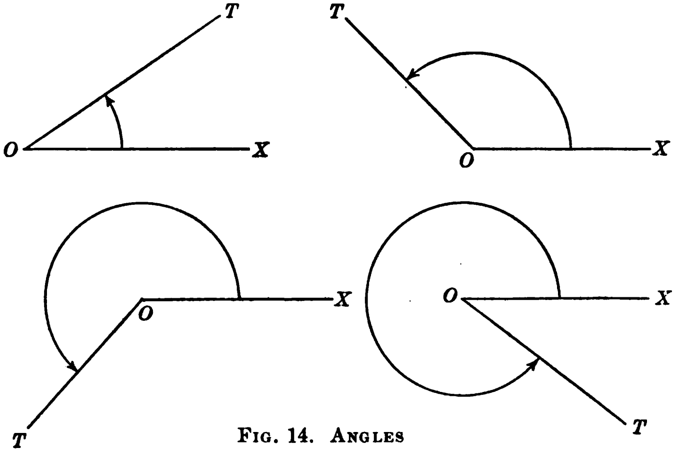


FIG. 14. ANGLES

reaches the position OT, it is said to turn through the *angle* XOT. Thus, *an angle is the amount of turning made by a line rotating in a plane about a fixed point.* Note that as the rotation continues the angle increases. Rotation is customarily considered as counterclockwise unless otherwise specified.

23. Vertex; initial side; terminal side. The fixed point O (Fig. 14) is called the *vertex* of the angle. The line OX is called the *initial side* of the angle, and the line OT is called the *terminal side* of the angle.

24. Symbols. The symbol for "angle" is $\angle$, and that for "angles" is $\angle$s. Thus, "angle XOT" is written $\angle XOT$.

25. Size of angles. From the definition of an angle given in Art. 22 we see that it is possible for the line OX (Fig. 14) to stop rotating so that the angle may contain any given amount of rotation (turning). The amount of rotation, therefore, and not the length of the sides determines the size of an angle. Thus, two angles are equal when they each contain the same amount of rotation; otherwise they are unequal. Since the sides may be extended indefinitely, an angle may have short or long sides, as $\angle A$ and $\angle B$ in Fig. 15, where $\angle A$ is greater than $\angle B$, but the sides of $\angle B$ are longer than the sides of $\angle A$.

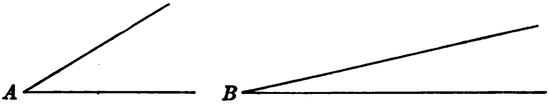
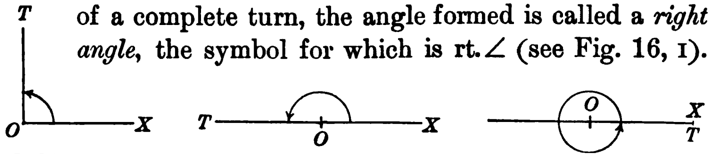


FIG. 15

26. Right angle; straight angle; perigon. If a line rotates in a plane about a fixed point so as to make one fourth of a complete turn, the angle formed is called a *right angle*, the symbol for which is rt.$\angle$ (see Fig. 16, I).



I. Right Angle II. Straight Angle III. Perigon

FIG. 16. THREE SPECIAL ANGLES

If the line makes one half of a complete turn, the angle formed is called a *straight angle*, the symbol for which is st.$\angle$

(see Fig. 16, II); if the line makes a complete turn, the angle formed is called a *perigon* (see Fig. 16, III).

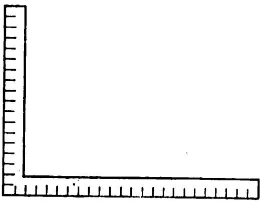


FIG. 17

Carpenters draw right angles by means of a tool called a *square*, which has two legs at right angles to each other (Fig. 17). The legs are graduated to fractions of an inch.

EXERCISE

Show why all straight angles must be equal; why all right angles must be equal.

27. Acute angle; obtuse angle; oblique angle; reflex angle. Angles are further classified by their relation to the right angle, the straight angle, and the perigon. An angle less than a right angle is called an *acute angle* (Fig. 18); one greater than a right angle but less than a straight angle is called an *obtuse angle* (Fig. 19). Any angle not a right angle is called an *oblique angle*. An angle greater than a straight angle but less than a perigon is called a *reflex angle* (Fig. 20). You can probably find examples of these angles in the classroom.

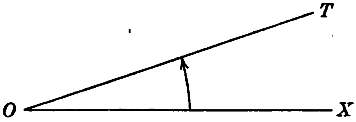


FIG. 18. ACUTE ANGLE

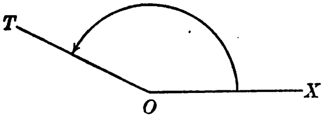


FIG. 19. OBTUSE ANGLE

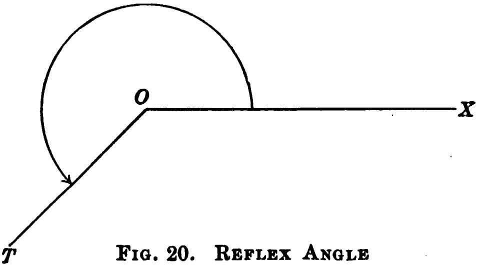


FIG. 20. REFLEX ANGLE

28. Notation for reading angles. There are three common methods by which we denote angles: (1) Designate the angle formed by the lines OX and OT (Fig. 21) as the "angle XOT" or the "angle TOX." Here the first and last letters denote points on the lines forming the angle, and the middle letter denotes the point of intersection (the vertex). In reading "angle XOT" we regard OX as the initial side and OT as the terminal side. (2) Designate the angle by a small letter, as x, placed as in Fig. 22. In writing equations this method is the most convenient. (3) Designate the angle by the letter which is written at the point of intersection of the two sides of the angle, as the "angle A" (called the *included angle* between AC and AB) (Fig. 22). This last method is used only when there is no doubt as to what angle is meant.

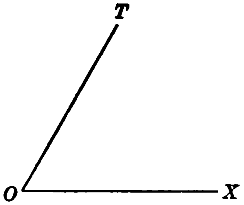


FIG. 21

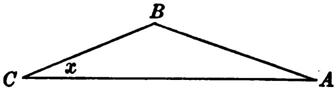


FIG. 22

29. Interior angles of a triangle. The angles at A, B, and C (Fig. 22) are called the *interior angles* of the $\triangle ABC$. The symbol for "triangle" is $\triangle$; that for "triangles" is $\triangle\!\!\!\triangle$.

30. Exterior angles of a triangle. If the three sides of a triangle are extended, one at each vertex, as in Fig. 23, the angles thus formed (x, y, and z) are called the *exterior angles* of the $\triangle ABC$.

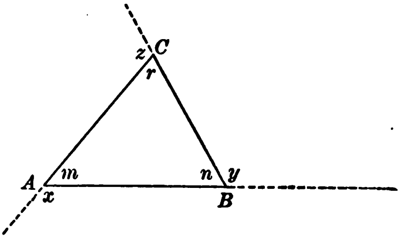


FIG. 23. ILLUSTRATING THE EXTERIOR ANGLES OF A TRIANGLE

EXERCISES

1. How many exterior angles can be drawn at each vertex of a triangle?

2. How many interior angles does a triangle have? How many exterior angles?

31. Circle. If a line OX (see Fig. 24) be taken as the initial side of an angle and the line be rotated one complete turn (a perigon), any point, as P, on the line OX traces a curved line which we call a *circle.* The symbol for "circle" is ⊙. Thus, *a circle is a closed curve all points of which lie in the same plane and are equally distant from a fixed point.*

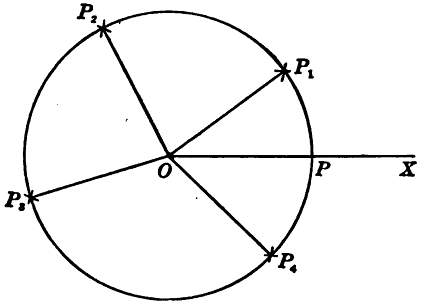


FIG. 24. THE CIRCLE

EXERCISE

Would the preceding definition be as good if we left out the phrase "lie in the same plane"? Explain your answer.

32. *Postulate.* *In a plane, from a given center and with a given radius, one circle and only one can be described.*

33. Center; circumference. The fixed point O (Fig. 24) is the *center* of the circle. The length of the circle is called the *circumference* (distance around) of the circle.

34. Radius; diameter. A line drawn from the center of a circle to any point on the circle is called a *radius.* Thus, OP in Fig. 24 is a radius of the circle. A line passing through the center of a circle and terminating on the circle is called a *diameter*.

From the definition of radius given above it is clear that in a given circle or in given equal circles one radius has the same length as any other radius. Thus we obtain the following important geometric relation: *Radii of the same circle or of equal circles are equal* ("radii" is the plural of "radius"). Likewise, *all diameters of the same circle or of equal circles are equal.*

35. Arc; central angle. An *arc* is a part of a circle. If two radii are drawn from the center of a circle to two different points on the circle, they cut off an arc on the circle. The symbol for "arc" is ⌒. Thus, $\widehat{AB}$ is read "arc AB." The angle formed by the radii at the center of the circle is called the *central angle.*

36. Quadrant; semicircle. An arc equal to one fourth of a circle is called a *quadrant*; an arc equal to one half of a circle is called a *semicircle.*

37. Measurement of angles; construction of angles; the protractor. In many instances the process of measuring angles is as important as that of measuring lines. An angle is measured by finding how many times it contains another angle selected as a unit of measure.

The *protractor* (Fig. 25) is an instrument for measuring and constructing angles. The protractor commonly consists of a semicircle divided into 180 equal parts. Each of these equal parts is called a *degree of arc* (the symbol for "degree" is °). In the measurement of angles we shall consider a unit called a *degree of angle.* Thus, to measure a given angle place the protractor so that the center of the protractor (O in Fig. 25) falls upon the vertex and make the straight edge of the protractor coincide with (fall upon) the initial side of the given angle. Now, observe where the terminal side of the given angle intersects (crosses) the rim of the protractor. Read the number of degrees in the

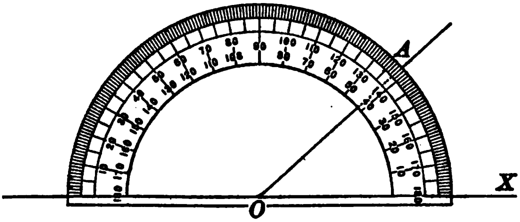


Fig. 25. Measuring Angles with a Protractor

angle from the scale on the protractor. The protractor is also useful in constructing angles of a required size. For example, to construct an angle of 42° draw a straight line OX (Fig. 25) and place the straight edge of the protractor on the line OX so that the center rests at O. Count 42° from the point on OX where the curved edge touches OX and mark the point A. Connect A and O, and the angle thus formed is an angle of 42°.

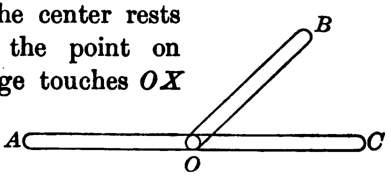


Fig. 26

Fig. 26 shows an instrument which can be used for transferring angles; it consists of two sticks, one of which is fastened to the middle of the other by a pivot.

EXERCISES

1. Construct an angle of 25°; of 37°; of 95°; of 68°; of 112°; of 170°. Continue this exercise until you are convinced that you can draw any required angle.

2. Construct an angle equal to a given $\angle ABC$ by means of the protractor.

38. Measuring angles out of doors. It is possible to measure angles out of doors by means of a simple *field* (out-of-door) *protractor*, so that some simple problems in surveying can be solved. A field protractor (see Fig. 27) may be made by a member of the class in the way described below.

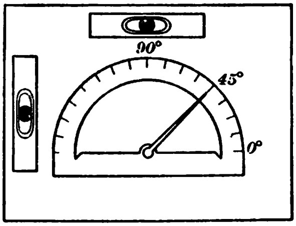


FIG. 27

Secure as large a protractor as possible and fasten it on an ordinary drawing board. Attach the board to a camera tripod (if this is not available, a rough tripod may be made). Make a slender pointer and attach it at the center of the protractor with a pin so that it shall swing freely about the pin as a pivot. Place two inexpensive carpenter's levels on the board, and the instrument is ready for use.

EXERCISES

1. The angle of elevation of the sun above the horizon, the $\angle SAB$ in Fig. 28, may be found as follows: Hold a protractor as shown, so that the plumb line AC, which is suspended from a large pin at A, falls upon the 90° mark on the protractor. The shadow of the pin will then fall upon the protractor at E. The reading of the $\angle DAE$ will then be the angle of elevation of the sun. Explain why this is true.

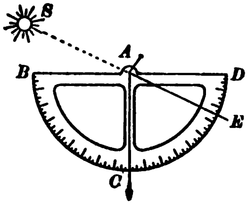


FIG. 28

2. An instrument known as a *quadrant* was probably first used about the beginning of the seventeenth century for measuring the altitudes of the stars. When the quadrant is held in a vertical position, as shown in Fig. 29, with the points A and C in a line with the star S, the angle of elevation of the star is given by the $\angle BAD$. Explain why this is true.

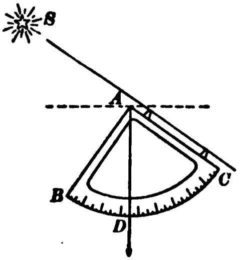


FIG. 29

39. Adjacent angles; exterior sides. In Fig. 30 $\angle x$ and $\angle y$ are two angles which have a common vertex and a common side between them, and are said to be *adjacent angles.* Thus, adjacent angles are angles that have the same vertex and a common side between them. The sides OT and OR are called the *exterior sides.*

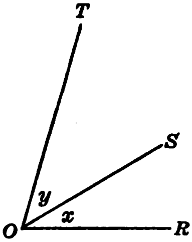


FIG. 30

40. Geometric addition and subtraction of angles. In adding two given angles, such as x and y in Fig. 31, $\angle y$ is placed adjacent to $\angle x$, and the resulting angle is called the sum of x and y. The angles may be transferred to the new position either by means of tracing paper or, more conveniently, by means of the protractor.

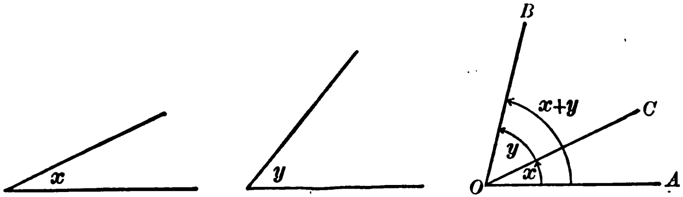


FIG. 31. GEOMETRIC ADDITION OF ANGLES

We may also find the difference between two angles. In Fig. 32 the two given angles are x and y. Place the smaller angle (y) on the larger angle (x) so that the vertices and one side of each angle coincide. The part remaining between the other sides of x and y will be the difference between x and y. Thus, in Fig. 32 we have $\angle x - \angle y = \angle AOC$.

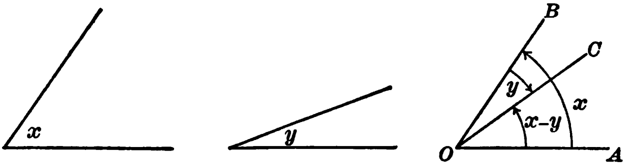


FIG. 32. GEOMETRIC SUBTRACTION OF ANGLES

EXERCISES

1. Add two angles by placing them adjacent to each other.

2. Draw three unequal angles, designating them x, y, and z so that $y > x$ and $x > z$. Draw an angle equal to $x + y + z$; equal to $y - x + z$; equal to $y + x - z$.

41. *Construction problem.* *At a given point on a given line to construct by means of ruler and compasses an angle equal to a given angle.*

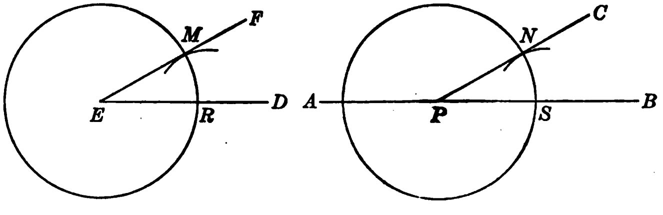


FIG. 33. CONSTRUCTION OF AN ANGLE EQUAL TO A GIVEN ANGLE

Construction. Let DEF in Fig. 33 be the given angle and let P be the given point on the given line AB.

With E as a center and ER as a radius draw a circle. With P as a center and with the same radius (ER) draw another circle cutting

AB at S. Place the sharp point of the compasses at R and draw an arc that shall pass through M. With S as a center and the same radius (RM) draw an arc cutting the second circle at N. Draw PC through N.

The $\angle BPC$ is the required angle.

The proof of this construction is given in Art. 138.

In this construction we make use of the following simple geometric relation between central angles of a circle and their arcs: *In the same circle or in equal circles equal central angles have equal arcs on the circle.* For example, if the central angle contains 19° of angle, then the arc contains 19° of arc.

It is easy to see that the following geometric relation is also true: *In the same circle or in equal circles equal arcs on the circle have equal central angles.*

The two preceding geometric relations make clear why the protractor may be used to measure angles, as was described in Art. 37. The method used there is based upon the idea that every central angle of 1° has an arc of 1° on the rim of the protractor. That is, *a central angle is measured by its arc on the circle* when degrees of angle and degrees of arc are used as the respective units of measure.

EXERCISES

1. Construct an angle equal to three times a given angle.

2. Construct two angles, designating one of them as containing a degrees and the other as containing b degrees. Using these angles, construct an angle containing $a + b$ degrees; an angle containing $2a + b$ degrees.

3. Choose a and b in Ex. 3 so that $a > b$, and construct an angle equal to the difference of the two given angles.

4. Construct an angle equal to the sum of three given angles.

42. Function. From the fact that a central angle is measured by its arc we say that the size of the central angle depends upon the size of the arc if the circle remains constant (the same size); or we say that the central angle is a *function* of its arc.

43. Perpendicular. We have seen in Art. 26 how right angles are formed by rotation. If two lines form right angles with each other, they are said to be *perpendicular* to each other. The symbol for "is perpendicular to" is ⊥.

44. *Construction problem.* *At a given point on a given line to erect a perpendicular to the line by means of ruler and compasses.*

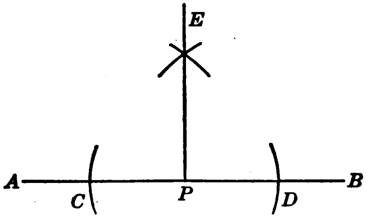


FIG. 34. HOW TO ERECT A PERPENDICULAR

Construction. Let AB be the given line and P the given point (Fig. 34). With P as a center and with a convenient radius draw arcs intersecting AB at C and D.

With C and D as centers and with a radius greater than $\frac{1}{2}$ CD draw arcs intersecting at E. Draw EP.

Then EP is the required perpendicular.

NOTE. For the present we shall assume that *at a point on a given line one and only one perpendicular to the line can be erected.*

EXERCISES

1. Why must the radius CE used in Fig. 34 be greater than $\frac{1}{2}$ CD?

2. In Fig. 34 how would you proceed to erect a perpendicular to the line AB at the point B instead of at the point P?

45. *Construction problem.* *To bisect a given line segment AB.*

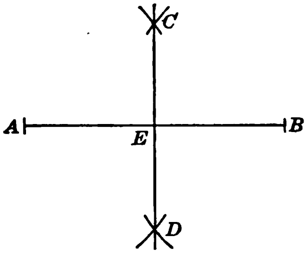


FIG. 35. HOW TO BISECT A LINE SEGMENT

Construction. Let AB be the given line segment (Fig. 35). With A as a center and with a radius greater than $\frac{1}{2}$ AB describe arcs above and below AB. With B as a center and with the same radius as before describe arcs above and below AB and intersecting the first arcs at C and D. Draw CD.

Then E is the point of bisection for AB.

46. Perpendicular bisector. The line CD in Fig. 35 is called the *perpendicular bisector* of the line AB.

EXERCISES

1. How may a line be divided into four equal parts? into eight equal parts?

2. Can the method of bisecting a line segment given in Art. 45 be used to divide a line segment into six equal parts? Explain your answer.

3. In general, into how many equal parts can a line segment be divided by the method of Art. 45?

4. Draw a triangle all of whose angles are acute (an acute-angled triangle). Construct the perpendicular bisector of each of the three sides of the triangle.

5. Draw a triangle in which one angle is obtuse (an obtuse-angled triangle) and draw the perpendicular bisectors of the three sides.

6. Draw a triangle in which one angle is a right angle and construct the perpendicular bisectors of the sides.

7. Draw a triangle, bisect each side, and connect each point of bisection with the opposite vertex.

47. Median. A line joining the vertex of a triangle to the midpoint of the opposite side is called a *median.*

48. ***Construction problem.*** *From a given point outside a given line to drop a perpendicular to the line.*

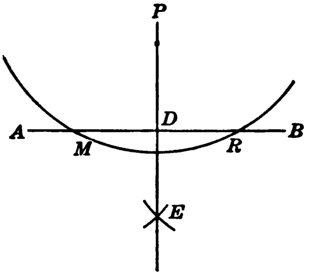


FIG. 36. HOW TO DROP A PERPENDICULAR

Construction. Let AB be the given line and P the given point (Fig. 36). With P as a center and with a radius greater than the distance from P to AB describe an arc cutting AB at M and R. With M and R as centers and with a radius greater than $\frac{1}{2}$ MR describe arcs either above or below (preferably below) the line AB. Connect the point of intersection E with P.

Then the line PD is perpendicular to AB, as required. Test the accuracy of your work by measuring an angle at D.

NOTE. For the present we shall assume that *through a point not on a given line one and only one perpendicular can be drawn to the line.*

EXERCISE

Draw a triangle all of whose angles are acute, and draw a perpendicular from each vertex to the opposite side.

49. Altitude. An *altitude* of a triangle is a line drawn from a vertex perpendicular to the opposite side.

EXERCISES

1. Draw a triangle in which one angle is obtuse and draw the three altitudes.

2. Draw a triangle in which one angle is a right angle and draw the three altitudes.

3. When do the altitudes fall inside a triangle? outside a triangle?

50. Bisector of an angle. Suppose $\angle ABC$ to be a given angle (Fig. 37). With the vertex B as a center and with a convenient radius draw an arc cutting BA and BC at X and Y respectively. With X and Y as centers and with a radius greater than $\frac{1}{2}$ XY draw arcs meeting at D. Join B and D. Then BD is the *bisector* of the $\angle ABC$.

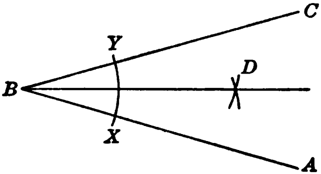


FIG. 37. HOW TO BISECT AN ANGLE

EXERCISES

1. Bisect an angle of 30°; of 45°; of 60°; of 90°.

2. Divide a given angle into four equal parts.

3. Can the above method of bisecting an angle be used to divide an angle into six equal parts? Explain your answer.

4. In general, into how many parts can an angle be divided by the method given above?

5. Draw a triangle whose angles are all acute, and bisect each of the angles.

51. Parallel lines. The lines AB and CD in Fig. 38 have the same amount of angular rotation from the initial line

EF. Thus they have the same direction and are said to be *parallel*. The symbol for "is parallel to" is ∥. Thus, $AB \parallel CD$ is read "*AB* is parallel to *CD*."

52. Corresponding angles; transversal. In Fig. 38 ∠s x and y are called *corresponding angles*. These angles are sometimes called *exterior-interior angles*. The line *EF* is called a *transversal*. It is clear that the lines are parallel only when the corresponding angles are equal and that the corresponding angles are equal only when the lines are parallel.

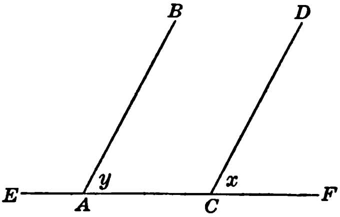


FIG. 38. PARALLEL LINES

EXERCISES

1. Draw figures to illustrate the importance of the last statement above.

2. Point out the parallel lines which you can find in the classroom.

53. *Construction problem.* *To draw a line parallel to a given line.*

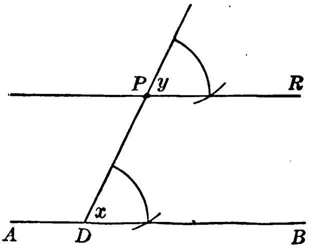


FIG. 39. HOW TO DRAW PARALLEL LINES

Construction. Choose a point P outside the given line *AB* in Fig. 39. Draw a line through P so as to form a convenient angle (x)

with AB. Call the point of intersection D. At P, using DP as the initial line, construct a corresponding angle (y) equal to the $\angle x$ by the method of Art. 41.

Then PR and AB are parallel because they have the same amount of rotation from the initial line PD.

54. Postulate. *Through a given point not on a given straight line one and only one straight line can be drawn parallel to the given line.*

EXERCISES

1. Construct a line parallel to a given line through a given point outside the line.

2. A carpenter wishes a board, one edge of which is straight, to have parallel ends. He makes a mark across each end with his square and saws along the marks. Why are the ends parallel?

3. Two parallel lines are cut by a transversal so as to form two corresponding angles one of which is $(x + 125°)$ and the other of which is $(3x + 50°)$. Find x and the size of each angle. Make a drawing to illustrate your work.

4. If $AB \parallel CD$ in Fig. 40, what other angles besides x and y are corresponding angles?

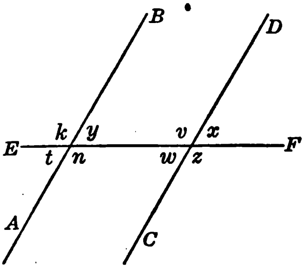


FIG. 40

5. In Fig. 40 $\angle x = \angle y$. Bisect $\angle x$ and $\angle y$ and show that these bisectors are parallel to each other.

55. Parallelogram. If one pair of parallel lines cross (intersect) another pair, the four-sided figure thus formed is called a *parallelogram*; that is, *a parallelogram is a quadrilateral the opposite sides of which are parallel.*

EXERCISE

Point out as many examples of the use of parallelograms as you can.

56. Constructing a parallelogram. If we remember the method used in Art. 53 for constructing one line parallel to another, it will be easy to construct a parallelogram. First, draw a working line *AB* (Fig. 41) and draw *AR*, making a convenient angle with *AB*. Through any point, as *P*, on *AR* draw a line *PV* parallel to *AB*. Through any point *M* on *AB* draw a line *MT* parallel to *AR*. The figure *AMSP* is a parallelogram, since its opposite sides are parallel.

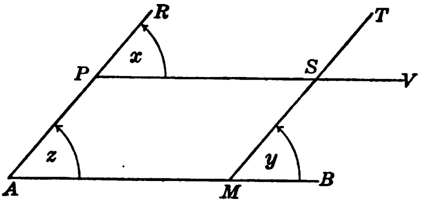


FIG. 41. HOW TO CONSTRUCT A PARALLELOGRAM

57. Rectangle. If one of the interior angles of a parallelogram is a right angle, all the angles are right angles, and the figure is a *rectangle* (Fig. 42). Thus, *a rectangle is a parallelogram in which the interior angles are right angles.*

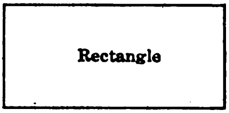


FIG. 42

EXERCISE

Show that all the angles formed by the sides of a rectangle extended are right angles.

58. Square. If all the sides of a rectangle are equal, the figure is called a *square* (Fig. 43).

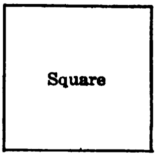


FIG. 43

59. Rhombus. A rhombus (Fig. 44) is a special case of a parallelogram. *A rhombus is a parallelogram all sides of which are equal and all angles of which are oblique.*

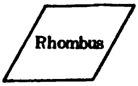


FIG. 44

EXERCISES

1. Give examples of rectangles; of squares.

2. Construct a rectangle having two adjacent sides equal to 5 cm. and 8 cm. respectively (use compasses and straightedge only).

3. Construct a rectangle having the two adjacent sides equal to the line segments *a* and *b* in Fig. 45.

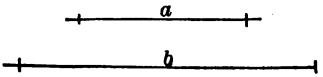


FIG. 45

4. Construct a square 7 cm. on a side.

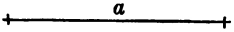


FIG. 46

5. Construct a square a side of which is *a* units long (use the line *a* in Fig. 46).

6. Given one side, construct a rhombus.

60. Left side of an angle; right side of an angle. If in Fig. 47 we imagine ourselves standing at the vertex of the $\angle ABC$ and looking in the direction BD, then the side BC is called the *left side* of the angle and the side BA is called the *right side* of the angle.

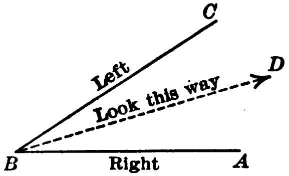


FIG. 47

61. Notation. When lettering angles and figures we often denote angles or lines that have certain characteristic likenesses by the same letter so as to identify them more easily. To use l for the left side of one angle and also for the left side of another angle in the same discussion would be misleading. In order to be clear, therefore, we let l_1 stand for the left side of one angle, l_2 for the left side of a second angle, and l_3 for the left side of a third angle, and so on. Then the sides are read "l sub-one," "l sub-two," "l sub-three," and so on.

62. Angles with parallel sides. The two angles (x_1 and x_2) in Fig. 48 are drawn so that their sides are parallel left to left and right to right. How do they seem to compare in size? Check your estimate by measuring with a protractor. Discuss the relation and show that $x_1 = x_2$.

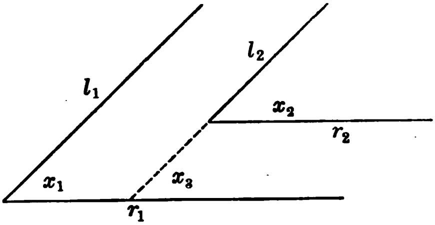


FIG. 48

It is evident that *if two angles have their sides parallel left to left and right to right, the angles are equal.*

EXERCISE

Draw freehand two obtuse angles such that their sides look parallel left to left and right to right. The angles should be approximately equal. Are they?

HINT. Take two points for vertices and in each case imagine yourself standing at the point. Draw the left sides to your left and the right sides to your right. Assume that the drawing is correct and prove that the angles are equal.

63. Angles with parallel sides. The two angles (x and y) in Fig. 49 have been drawn so that their sides are parallel left to right and right to left. What relation seems to exist between them? Measure each angle with a protractor. Discuss the relation and show that $x + y = 180°$.

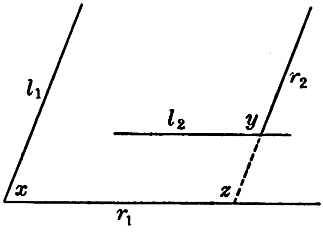


FIG. 49

From a proper discussion the student should see that *if two angles have their sides parallel left to right and right to left, their sum is a straight angle.*

EXERCISE

Practice drawing freehand a pair of angles whose sides are parallel according to the conditions above. Is the sum approximately 180°?

64. Supplementary angles; supplement. Two angles whose sum is equal to a straight angle (180°) are said to be *supplementary angles.* Each angle is called the *supplement* of the other.

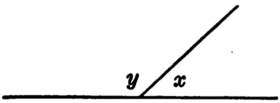


FIG. 50

65. Supplementary adjacent angles. Place two supplementary angles adjacent to each other as in Fig. 50. Angles so placed are called *supplementary adjacent angles.* Their exterior sides form a straight line.

EXERCISES

1. In Fig. 50 what is the angle whose supplement is $\angle x$?

2. How many degrees are there in the supplement of an angle of 30°? of 90°? of 150°? of x°?

66. Complementary angles. If the sum of two angles is a right angle, the two angles are called *complementary angles.* Each angle is called the *complement* of the other. Thus, in Fig. 51 $\angle x$ is the complement of $\angle y$.

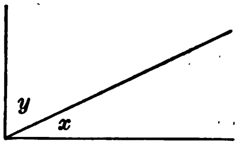


FIG. 51

EXERCISES

1. What is the complement of 30°? of 60°? of x°?

2. What is the relation existing between the exterior sides of two adjacent complementary angles?

67. Vertical angles. Draw two intersecting straight lines AB and CD as in Fig. 52. The $\measuredangle s$ x and z are called *vertical*, or *opposite, angles.* Notice that vertical angles have a common vertex and that their sides lie in the same straight line but in opposite directions. We may use this as the definition of vertical angles.

Are w and y in Fig. 52 vertical angles? How do they compare in size? The four exercises below show that *if two lines intersect, the vertical angles are equal.*

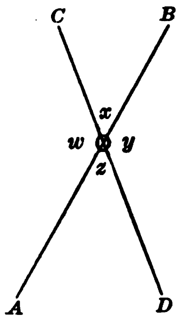


FIG. 52. VERTICAL ANGLES

EXERCISES

(The exercises refer to Fig. 52)

1. What is the sum of $\angle x$ and $\angle y$? of $\angle z$ and $\angle y$?

2. Show that $x + y = z + y$.

3. How does Ex. 2 help in obtaining the relation between x and z? What is this relation?

4. Show that $y + z = x + w$ and from this show that $y = w$.

68. Alternate-interior angles. In Fig. 53 the $\measuredangle s$ x and y, formed by the lines AB, CD and the transversal EF, are called *alternate-interior angles* (on alternate sides of EF and interior with respect to AB and CD).

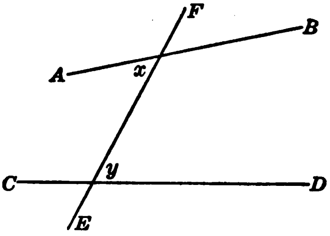


FIG. 53

EXERCISES

1. Show by reference to the definition of parallel lines in Art. 51 that *if two parallel lines are cut by a transversal, the corresponding angles are equal.*

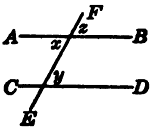


FIG. 54

2. Measure and compare $\angle x$ and $\angle y$ in Fig. 54.

3. In Fig. 54 the lines AB and CD are parallel. What is the relation between $\angle x$ and $\angle y$? Explain how you got your answer.

69. Theorem. The answer to Ex. 2 in Art. 68 suggests the following geometric relation: *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.* Such a statement of a geometric relation to be proved is called a *theorem.* We shall give a formal proof for this theorem in Art. 70.

70. ***Theorem.*** *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*

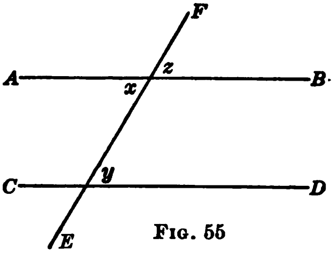


FIG. 55

Given AB parallel to CD and cut by the transversal EF.

To prove that $\angle x = \angle y$.

Proof

STATEMENTS	REASONS
1. $AB \parallel CD$.	1. Given.
2. Then $\angle y = \angle z$.	2. If two parallel lines are cut by a transversal, the corresponding angles are equal.
3. But $\angle x = \angle z$.	3. Vertical angles are equal.
4. $\therefore \angle x = \angle y$.	4. Because they are each equal to the $\angle x$.

NOTE. The symbol $\therefore$ is read "therefore."

71. Interior angles on the same side of the transversal. In Fig. 56 the $\angle$s x and y are called *interior angles on the same side of the transversal.*

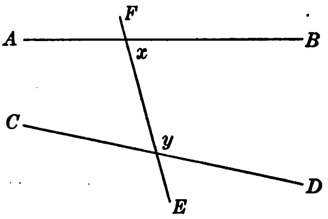


FIG. 56

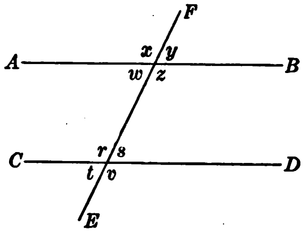


FIG. 57

EXERCISES

1. In Fig. 57 select all the pairs of corresponding angles, alternate-interior angles, and interior angles on the same side of the transversal.

2. Prove that *if two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

72. Alternate-exterior angles. In Figs. 58 and 59 the $\angle$s b and h and the $\angle$s a and g are called *alternate-exterior angles.*

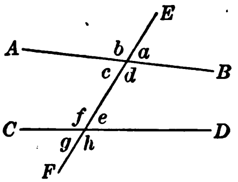


FIG. 58

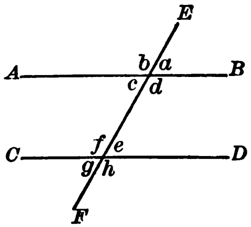


FIG. 59

EXERCISES

1. In Fig. 59 select all the pairs of alternate-exterior angles.

2. *If two parallel lines are cut by a transversal, the alternate-exterior angles are equal.*

NOTE. In the statement of exercises we shall often omit the words "prove that" or "show that" where it is obvious that a proof is required.

3. Two parallel lines are cut by a transversal so as to form angles as shown in Fig. 60. Find x and the size of each of the eight angles in the figure.

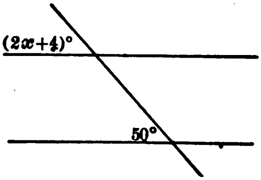


FIG. 60

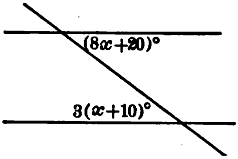


FIG. 61

4. Find x and the size of each of the eight angles in Fig. 61.

5. Draw two parallel lines and a transversal. Select all the pairs of equal angles; all the pairs of supplementary angles.

73. Outline of angle pairs formed by two lines cut by a transversal. When two lines are cut by a transversal, as in Figs. 62, I, and 62, II,

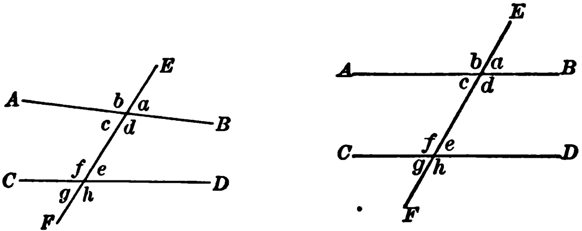
$$\begin{matrix}\text{the angles of the}\\ \text{angle pairs}\end{matrix}\left\{\begin{matrix} a \text{ and } e\\ b \text{ and } f\\ d \text{ and } h\\ c \text{ and } g\end{matrix}\right\}\text{are called } \textit{corresponding angles};$$

the angles c, d, e, f are called *interior angles*;
the angles a, b, g, h are called *exterior angles*;

$$\begin{matrix}\text{the angles of the}\\ \text{angle pairs}\end{matrix}\left\{\begin{matrix} d \text{ and } e\\ c \text{ and } f\end{matrix}\right\}\begin{matrix}\text{are called } \textit{interior angles on the}\\ \textit{same side of the transversal};\end{matrix}$$

the angles of the angle pairs $\left\{\begin{array}{l} d \text{ and } f \\ c \text{ and } e \end{array}\right\}$ on opposite sides of the transversal are called *alternate-interior angles*;

the angles of the angle pairs $\left\{\begin{array}{l} b \text{ and } h \\ a \text{ and } g \end{array}\right\}$ on opposite sides of the transversal are called *alternate-exterior angles.*



I II

FIG. 62

The student should remember that

(a) corresponding angles are equal,
(b) alternate-interior angles are equal,
(c) alternate-exterior angles are equal,
(d) interior angles on the same side of the transversal are supplementary,

only when the lines cut by the transversal are parallel (see Fig. 62, II).

EXERCISES

(The exercises refer to Fig. 63)

1. Measure with the protractor and compare $\angle x$ and $\angle y$.

2. The lines AB, CD, and FE are drawn so that $\angle x = \angle y$. What seems to be the relation between the lines AB and CD?

3. Show that if $\angle x = \angle y$, then $\angle y = \angle z$.

4. Show that AB is parallel to CD (see the definition for parallel lines, Art. 51).

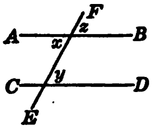


FIG. 63

74. Theorem. *If the alternate-interior angles formed by two lines and a transversal are equal, the lines are parallel.*

Proof

STATEMENTS	REASONS
1. $\angle x = \angle y$ (see Fig. 63).	1. Given.
2. $\angle x = \angle z$.	2. Vertical angles are equal.
3. Then $\angle y = \angle z$.	3. Because they are each equal to the $\angle x$.
4. $\therefore AB \parallel CD$.	4. By definition of parallel lines.

EXERCISES

1. In Fig. 64 construct a line through P parallel to the line AB by making an alternate-interior angle equal to $\angle x$. Show why the lines are parallel.

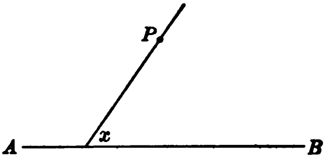


FIG. 64

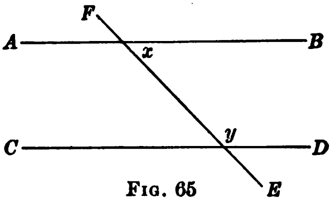


FIG. 65

2. Measure the ∠s x and y in Fig. 56 and find their sum.

3. In Fig. 65 the lines are drawn so that $\angle x + \angle y = 180°$. What seems to be the relation between the lines AB and CD?

4. *If the interior angles on the same side of a transversal between two parallel lines are supplementary, the lines are parallel.*

5. *If the alternate-exterior angles formed by two lines and a transversal are equal, the lines are parallel.*

6. Taking any two points on a given straight line, draw a perpendicular at each point. Show that *if two straight lines are perpendicular to the same line, they are parallel* (see Fig. 66).

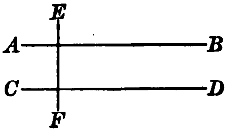


FIG. 66

7. The instrument shown in Fig. 67 is called a *T-square*, and is used by draftsmen for drawing parallel lines. The draftsman slides the T-square along the edge of a drafting board and rules lines along the arm of the T-square. Why are the lines parallel?

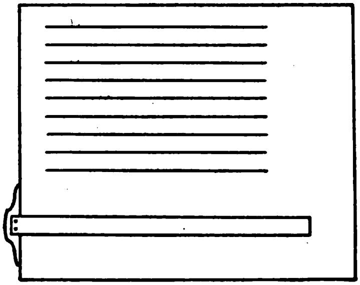


FIG. 67. T-SQUARE

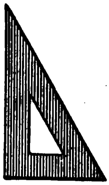


FIG. 68. WOODEN TRIANGLE

NOTE. A draftsman also uses what is called a *draftsman's triangle* (Fig. 68) for drawing parallel lines.

75. The student should remember that two straight lines cut by a transversal are parallel only when

(a) the corresponding angles are equal,
(b) the alternate-interior angles are equal,
(c) the alternate-exterior angles are equal,
(d) the interior angles on the same side of the transversal are supplementary.

76. ***Theorem.*** *The sum of the interior angles of any triangle is 180°.*

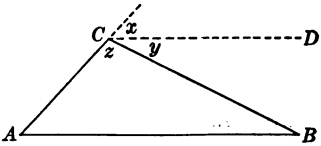


FIG. 69

Given the triangle *ABC*.

To prove that $\angle A + \angle B + \angle C = 180°$.

Proof

STATEMENTS	REASONS
1. Draw $CD \parallel AB$.	1. Through a point not on a given line one and only one line may be drawn parallel to the given line.
2. Then $\angle x = \angle A$.	2. Corresponding angles formed by two parallel lines cut by a transversal are equal.
3. And $\angle y = \angle B$.	3. Alternate-interior angles formed by two parallel lines cut by a transversal are equal.
4. But $\angle x + \angle y + \angle z = 180°$.	4. The sum of all the angles about a point in a plane on one side of a straight line is 180°.
5. $\therefore \angle A + \angle B + \angle C = 180°$.	5. By substituting $\angle A$ for $\angle x$, $\angle B$ for $\angle y$, and $\angle C$ for $\angle z$.

NOTE. This is a formal proof of the theorem, since it is independent of any measurement of the angles.

EXERCISE

Write an equation which expresses the number of degrees in the sum of the angles of a triangle.

77. Practical use of the theorem "The sum of the interior angles of a triangle is a straight angle." The equation found in the exercise above is a very useful one, as it enables us to find one angle of a triangle when the other two are known. Thus, if we know that two angles of a triangle are 50° and 70°, we know that the third angle is 60°. This is of great practical value to the surveyor, who is thus enabled to find the size of all three angles of a triangle by measuring only two directly.

In the exercises that follow, the pupil will need to apply the theorem proved in the preceding article.

EXERCISES

In the following exercises

(a) Draw the triangle freehand.

(b) Denote the angles properly as given.

(c) Using the theorem of Art. 76, write down the equation representing the conditions of the exercise.

(d) Solve the equation and find the size of each angle.

(e) Check your solution by the conditions of the exercise.

1. The angles of a triangle are x, $2x$, and $3x$. Find x and the number of degrees in each angle.

2. The first angle of a triangle is twice the second, and the third is three times the first. Find the number of degrees in each angle.

3. If the three angles of a triangle are equal, what is the size of each angle?

4. Find the angles of a triangle if the first angle is twice the second, and the third is 15° less than twice the first.

5. The angles of a triangle are to each other as 1, 2, 3. What is the size of each angle?

HINT. Let $x =$ the first, $2x$ the second, and $3x$ the third.

6. Find the angles of a triangle if the first is $2\frac{1}{2}$ times the second increased by 10°, and the third is $\frac{1}{4}$ of the second.

7. In a triangle one angle is a right angle; the other two angles are represented by x and $\frac{1}{4}x$ respectively. Find the size of each angle.

8. How many right angles may a triangle have? How many obtuse angles? What is the greatest number of acute angles that a triangle may have? the least number?

9. Two angles (x and y) of one triangle are equal respectively to two angles (m and n) of another triangle. Show that the third angle of the first triangle equals the third angle of the second triangle.

78. *Theorem.* By solving Ex. 9 of Art. 77 we obtain the following theorem: *If two angles of one triangle are equal respectively to two angles of another triangle, the third angle of the first triangle is equal to the third angle of the second triangle.*

EXERCISES

1. Show that any two angles which have their sides perpendicular left to left and right to right are equal (see Fig. 70).

SUGGESTION. In Fig. 70 $l \perp l'$, and $r \perp r'$. Reproduce the angles on paper.

2. Show that any two angles which have their sides perpendicular left to right and right to left are supplementary.

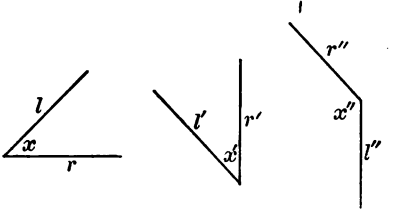


FIG. 70

SUGGESTION. In Fig. 70 $l \perp r''$, and $r \perp l''$. Reproduce the angles on paper.

79. Right triangle. If one angle of a triangle is a right angle, the triangle is called a *right triangle.* The symbol for "right triangle" is rt.△.

EXERCISES

1. Show that the sum of the acute angles of a right triangle is equal to a right angle.

2. What is the size of the other acute angle of a right triangle if one acute angle is 23°? if one acute angle is 45°?

80. Isosceles triangle; base angles. A triangle which has two equal sides is called an *isosceles triangle.* The angles opposite the equal sides are called the *base angles* of the isosceles triangle.

81. Theorem. *The sum of the exterior angles of a triangle* (taking one at each vertex) *is 360°* (2 st. ∠s).

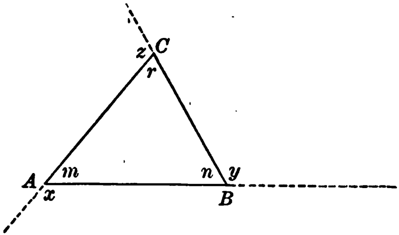


FIG. 71

Given the triangle *ABC*.

To prove that $\angle x + \angle y + \angle z = 360°$.

Proof

STATEMENTS	REASONS
1. $\angle x + \angle m = 180°$, $\angle y + \angle n = 180°$, and $\angle z + \angle r = 180°$.	1. The sum of the angles about a point on one side of a straight line is 180°.
2. $\therefore \angle x + \angle m + \angle y + \angle n + \angle z + \angle r = 540°$.	2. Equals added to equals give equals.
3. Then we may write the preceding equation $(\angle x + \angle y + \angle z) + (\angle m + n + \angle r) = 540°$.	3. By grouping the terms, since the sum is the same no matter what order the addends are in.
4. But $\angle m + \angle n + \angle r = 180°$.	4. The sum of the interior angles of a triangle is 180°.
5. $\therefore \angle x + \angle y + \angle z = 360°$.	5. Equals subtracted from equals give equals.

EXERCISES

1. Find the size of each interior and exterior angle in the triangle of Fig. 72.

2. Show that the exterior $\angle x$ of the $\triangle ABC$ in Fig. 73 is equal to the sum of the two nonadjacent interior ∠s A and C.

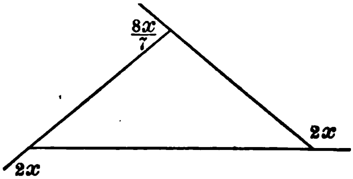


FIG. 72

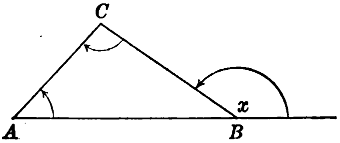


FIG. 73

3. In Fig. 74, $BD \parallel AC$. Prove that *an exterior angle of a triangle is equal to the sum of the two nonadjacent interior angles.*

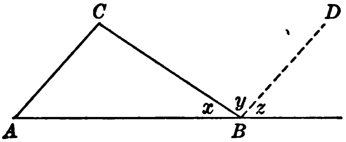
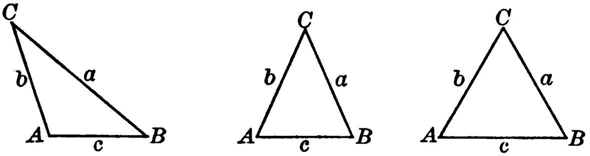


FIG. 74

4. Show that two different methods of proving Ex. 3 are suggested by Fig. 74.

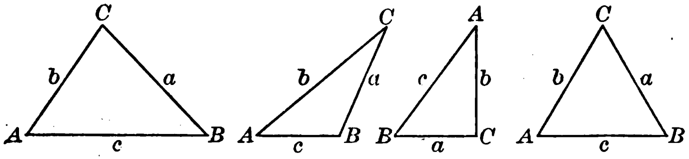
5. Prove Ex. 3 by drawing a line through C parallel to AB.

82. Classes of triangles. It is desirable for the student to be able to classify triangles properly. They are classified according to the relation of their sides and according to the kinds of angles, as is clearly shown in Figs. 75 and 76.



I. Scalene II. Isosceles III. Equilateral

FIG. 75. TRIANGLES CLASSIFIED ON BASIS OF RELATION OF SIDES



I. Acute II. Obtuse III. Right IV. Equiangular

FIG. 76. TRIANGLES CLASSIFIED ON BASIS OF ANGLES

83. Quadrilaterals. There are many interesting properties connected with the various kinds of quadrilaterals, as we shall find out in the following exercises:

EXERCISES

1. Draw a quadrilateral as in Fig. 77. Draw the diagonal AC dividing the quadrilateral into two triangles. What is the sum of the interior angles in each triangle? What, then, is the sum of the interior angles of a quadrilateral?

2. Draw a quadrilateral as in Fig. 77. Produce each side (one at each vertex). What do you estimate the sum of the exterior angles of the quadrilateral to be? Check your estimate by measuring the angles.

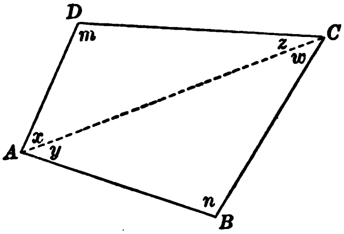


FIG. 77

3. *The sum of the exterior angles of a quadrilateral is 360°.*

4. Find the size of the interior angles of a quadrilateral in which each angle is 25° smaller than the consecutive angle.

5. *The consecutive interior angles of a parallelogram are supplementary*; that is, in Fig. 78 prove that $x + y = 180°$.

6. *The opposite interior angles of a parallelogram are equal.*

HINT. In Fig. 78 show that $\angle x = \angle a = \angle z$.

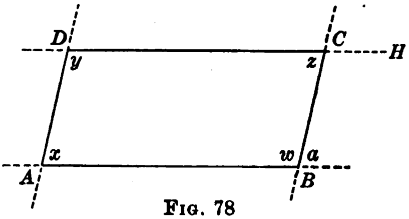


FIG. 78

7. If one interior angle of a parallelogram is twice as large as a consecutive angle, what is the size of each interior angle of the parallelogram?

8. The difference between two consecutive angles of a parallelogram is 30°. Find the size of each of the four interior angles in the parallelogram.

9. If one interior angle of a parallelogram is 20° more than three times a consecutive angle, what is the size of each interior angle of the parallelogram?

10. If one interior angle of a parallelogram is multiplied by 3 and this result increased by 30°, and the result obtained is 60° more than twice a consecutive angle, what is the size of each angle of the parallelogram?

84. Trapezoid. A *trapezoid* is a quadrilateral having only two sides parallel (see Fig. 79). Many practical uses are made of this geometric figure, as we shall see.

EXERCISES

1. *The sum of the interior angles of a trapezoid is two straight angles* (360°).

2. *Any two pairs of consecutive angles of a trapezoid are supplementary* (use Fig. 79).

3. In Fig. 79 $\angle D$ is 40° greater than $\angle A$ and $\angle B$ is 96° less than $\angle C$. Find the number of degrees in each angle.

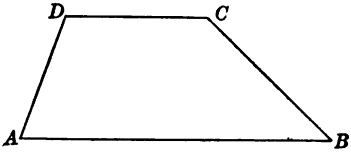


FIG. 79

85. Constructing triangles. We shall now proceed to study three constructions which require the putting together of angles and line segments into some required combination. Since these constructions are very important in all kinds of construction work, for example, in shop work, mechanical drawing, engineering, and surveying, the student should master them. Moreover, the student should be on the lookout for practical applications of these constructions in the world about him.

86. *Construction problem.* *To construct a triangle when two sides and the angle included between the sides are given.*

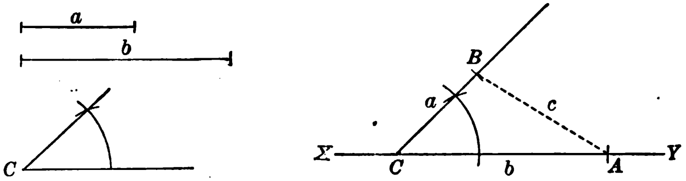


FIG. 80. HOW TO CONSTRUCT A TRIANGLE WHEN TWO SIDES AND THE INCLUDED ANGLE ARE GIVEN

Construction. Let the given sides be a and b and let the given angle be C, as shown in Fig. 80.

Draw a working line XY, and lay off $CA = b$. At C draw an angle equal to $\angle C$ by the method of Art. 41. With C as a center and with a radius equal to a lay off $CB = a$. Join B and A, and the triangle is constructed as required.

EXERCISES

1. Is the construction given above always possible?

2. Construct triangles when the given parts are as follows:

(a) $a = 3$ cm., $b = 4$ cm., $\angle C = 47°$.
(b) $c = 1$ in., $b = 2$ in., $\angle A = 112°$.
(c) $b = 1\frac{1}{2}$ in., $c = 1\frac{3}{4}$ in., $\angle A = 87°$.

3. Construct a triangle with the parts given in Fig. 81.

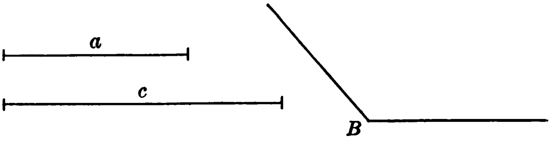


FIG. 81

4. Upon what does the length of the third side in Ex. 3 depend?

5. What determines the length of the third side in Ex. 3?

87. Construction problem. *To construct a triangle when two angles and the side included between them are given.*

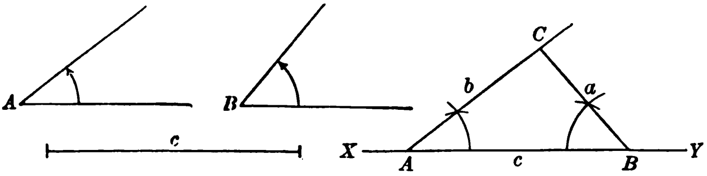


FIG. 82. HOW TO CONSTRUCT A TRIANGLE WHEN TWO ANGLES AND THE SIDE INCLUDED BETWEEN THEM ARE GIVEN

Construction. Let A and B be the given angles and let c be the given included side (Fig. 82).

Draw a working line XY, and lay off $AB = c$ on it. At A construct an angle equal to the given $\angle A$, at B construct an angle equal to the given $\angle B$, and produce the sides of those angles till they meet at C, as shown.

Then the $\triangle ABC$ is the required triangle.

EXERCISES

1. Construct triangles when the given parts are as follows:

(a) $\angle A = 30°$, $\angle B = 80°$, $c = 4$ cm.

(b) $\angle C = 110°$, $\angle B = 20°$, $a = 2$ in.

2. Draw a triangle with the parts given in Fig. 83.

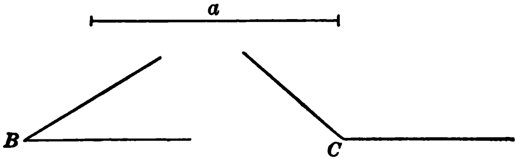


FIG. 83

3. Is the construction of Ex. 2 always possible?

4. Upon what do the lengths of b and c in Ex. 2 depend?

5. What determines the lengths of the sides b and c and the size of the $\angle A$ in Ex. 2?

88. *Construction problem.* *To construct a triangle when the three sides are given.*

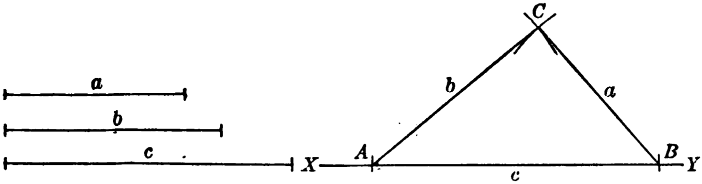


FIG. 84. HOW TO CONSTRUCT A TRIANGLE WHEN THREE SIDES ARE GIVEN

Construction. Let the given sides be a, b, and c, as shown in Fig. 84. Draw a working line XY, and lay off the side c, lettering it AB. With A as a center and with a radius equal to b construct an arc as shown. With B as a center and with a radius equal to a construct an arc intersecting the first arc at C, and join C to A and B.

Then the $\triangle ABC$ is the required triangle.

EXERCISES

1. Construct triangles with the following sides:

(a) $a = 5$ cm., $b = 5$ cm., $c = 8$ cm.
(b) $a = 7$ cm., $b = 8$ cm., $c = 4$ cm.
(c) $a = 7$ cm., $b = 9$ cm., $c = 3$ cm.

2. Is it always possible to construct a triangle when three sides are given?

3. Construct a triangle, using the sides given in Fig. 85.

4. A great deal of practical use is made of the fact that a triangle is a rigid figure; for example, a rectangular wooden gate is often divided into two triangles by means of a diagonal brace so as to make the gate more stable (less apt to sag or get out of shape). Give other examples of the practical use that is made of the stability of the triangular figure.

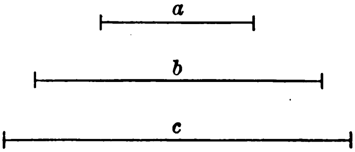


FIG. 85

5. Construct an isosceles triangle when the base and one of the two equal sides are given.

SUGGESTION. Use c in Fig. 85 as the base and use b twice; that is, in this case take $a = b$.

6. Construct an equilateral triangle when one of its sides is given.

89. Fundamental assumptions. The pupil should by this time be thoroughly familiar with the following fundamental assumptions, or *axioms*:

1. *Quantities equal to the same thing or to equal things are equal to each other.*
2. *In any function any quantity may be substituted for its equal.*
3. *If equals are added to equals, the sums are equal.*
4. *If equals are subtracted from equals, the remainders are equal.*
5. *If equals are multiplied by equals, the products are equal.*
6. *If equals are divided by equals, the quotients are equal* (provided, of course, the divisor is not zero).
7. *The whole is equal to the sum of its parts.*
8. *The whole is greater than any of its parts.*
9. *Like powers or like positive roots of equals are equal.*

CHAPTER II

CONGRUENCE OF RECTILINEAR FIGURES

90. Necessity of establishing truths by reasoning. No doubt the student has already observed that a number of geometric figures have been constructed, and that the accuracy of these constructions has been tested by measurement, tracings, or some other practical test. Many geometric truths are discovered in this quasi-experimental way.

We shall find it advantageous, however, to learn how to arrive at mathematical truths by a process of pure reasoning. The kind of reasoning we use in mathematics is the same as that which we use in everyday life whenever we infer one truth from another. In order to be good students of mathematics or of any other branch of knowledge we must learn to use certain principles and forms of thinking by means of which everybody must proceed in reasoning. For example, from the fact that we know that if a man is rich he can buy an automobile it is not safe to infer that if a man can buy an automobile he is rich. However, it is possible to infer from the fact that all men are mortal, and from the fact that Socrates was a man, that Socrates was mortal.

But there are always two dangers present in our thinking. First, we are often prone to draw wrong conclusions from the mere appearance of geometric figures (so-called optical illusions), and, second, we are often guilty of making false inferences or of being incorrect in our reasoning, because of insufficient data or of some other reason.

Logic is the only subject which treats of these principles; and although we cannot learn everything about logic in this course, we can at least try to develop rules of correct reasoning, in order that we may detect the weaknesses in an incorrect argument and make our own work more nearly perfect by developing sound reasoning powers.

There are many familiar illustrations of the danger of drawing false conclusions of one kind or another of which we shall present only a few of the simplest and best-known.

91. Optical illusions. Most people pride themselves on the fact that they can believe what they see with their own eyes. However, there are many instances where seeing should not necessarily be believing, because the eye is often a poor judge of the facts in the situation, especially if it is not supported by some kind of a check. Objects or figures which are so constructed as to give us the wrong idea about the facts are called *optical illusions.* The following exercises will help the student to see that he must be careful in making judgments merely from what he sees.

EXERCISES

1. Look first at one picture and then at the other in Fig. 86, and you will see that at one time there are six and at another time seven cubes. Can you explain the illusion ?

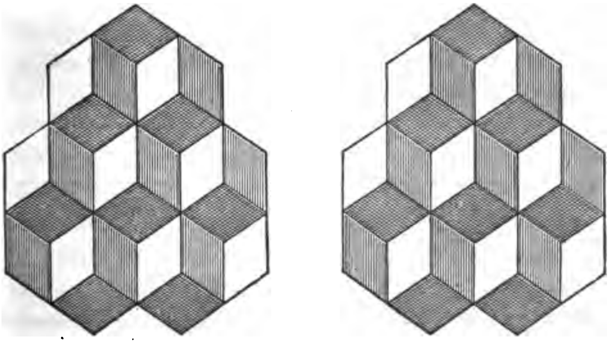


FIG. 86. OPTICAL ILLUSION

2. In Fig. 87 the light vertical line CD looks longer than the heavy horizontal line AB, but, as a matter of fact, they are equal in length. Check this statement by measurement.

NOTE. This exercise illustrates a well-known principle; namely, that a narrow object looks longer than a wide object of a similar kind and of the same length. This is why tall, thin people look taller than they really are, and why short, stout people look shorter than they really are. This principle should be taken into account in the selection of clothes. For example, a short, stout man should wear a suit with vertical stripes, but a tall, thin man should avoid them.

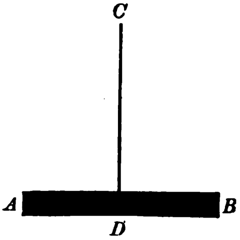


FIG. 87

3. Mention other applications of the principle illustrated by Ex. 2.

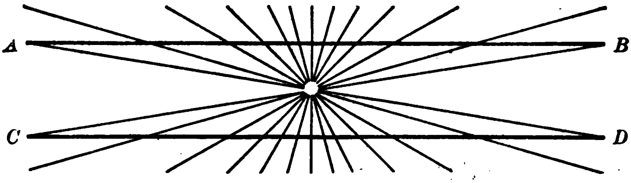


FIG. 88

4. The lines AB and CD in Fig. 88 are not only straight but parallel. Can you see this better by looking lengthwise at the lines from the ends or by using a ruler?

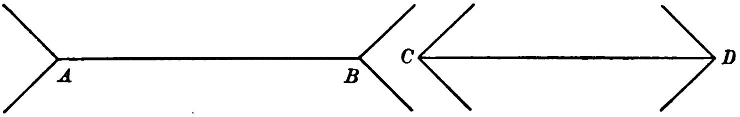


FIG. 89

5. Which of the lines in Fig. 89 is the longer? Show why your answer is correct.

6. In Fig. 90 the lower segment, π, looks larger than the upper segment, π', but as a matter of fact they have the same shape and the same size. Make a tracing of the two segments, cut them out, and show that they can be made to coincide.

7. Does Fig. 91 look like a square? Verify your answer by measurement.

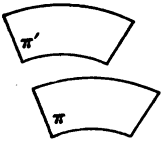


FIG. 90

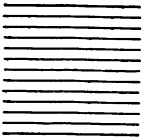


FIG. 91

8. In Fig. 92 the thin, narrow line running upward and to the right of the heavy black rectangle is the extension of one of the two lines extending downward and to the left. Look directly at the picture and see which line you think is extended upward. Then hold the figure so that you can look lengthwise along the lines, and see if it still appears the same. The line on the right seems to be the extension of the upper one on the left, but in reality it is the extension of the lower line.

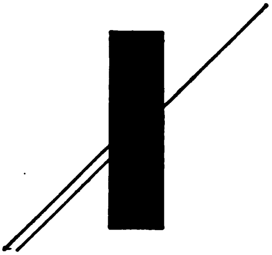


FIG. 92

92. Demonstrative geometry. The student has no doubt observed by this time that many facts concerning geometric figures can be discovered by careful drawings, observation, and measurement. Geometry treated by these methods is often called *intuitional geometry* or *experimental geometry.* But we have pointed out in the preceding paragraphs the limitations of these methods and the desirability of accepting statements only when their truth can be proved by pure reasoning, except in the case of a few fundamental truths which are assumed as a basis. The method of proving the truth of statements in this way is known as the *demonstrative method,* and geometry treated by this method is called *demonstrative geometry.*

93. When two geometric magnitudes are equal. We are often interested in comparing two geometric magnitudes of the same kind, as two linear distances, two angles, or the areas of two triangles. Whenever we can measure two magnitudes of the same kind and express their measures by the same number in terms of some common unit of measure, we say that the two magnitudes are equal.

It is often desirable in geometry, however, to compare two magnitudes of the same kind — say two line segments, two angles, two triangles, or two other geometric figures — without actually measuring them. Thus, in Art. 20 we showed that the line segment *a* was equal to the line segment *b* when the segments coincided throughout; and in Art. 21 we showed that if two line segments did not coincide throughout, the segments were not equal.

In Art. 25 we showed that the size of an angle is determined by the amount of rotation and not by the lengths of the sides of the angle. Hence we may determine that two angles are equal by finding that their sides fall upon each other. We need not show that they coincide throughout.

94. Superposition. The process of placing one geometric figure on another to see if the boundaries coincide is called *superposition*, and the one figure is said to be *superposed* on the other.

95. Congruence. When one geometric magnitude can be placed upon another so that the two coincide throughout, the geometric magnitudes are said to be *congruent.* Thus, *congruent figures are figures which can be made to coincide throughout.*

From the definition of congruence (the property of being congruent) it is clear that congruent figures are always equal, but equal figures are not necessarily congruent. In the case of line segments we know that if the segments are equal they are congruent, since we are concerned only with length. But in the case of geometric magnitudes having two or more dimensions, equality does not necessarily imply congruence.

Fig. 93 will help the student to understand the preceding paragraph. Thus, he can see that if two figures are equal they have the same area (size) but not necessarily the same shape (similar). On the other hand, if they are congruent they have the same size and the same shape.

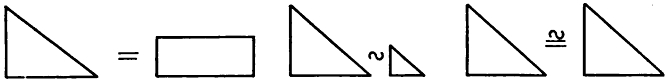


FIG. 93

NOTE. The symbols in Fig. 93 are explained in Art. 96.

Some writers use "equal" instead of "congruent" and "equivalent" instead of our word "equal," but we shall use only the meanings given in this text.

96. Congruent triangles. Triangles that have the same shape and the same size can be made to coincide throughout and are therefore called *congruent triangles.* The symbol for "is congruent to" is ≌, the ∽ indicating that the triangles have the same shape (similar), and the = indicating that they are the same size (equal in area). Thus, the statement $\triangle ABC \cong \triangle A'B'C'$ is read "Triangle ABC is congruent to triangle A-prime, B-prime, C-prime."

NOTE. The symbol ≌ was used in a modified form by Leibniz. The ∽ is probably a degenerate form of the letter S fallen over on its side. It is the initial letter of the Latin word "similis," meaning *similar.*

97. Experimental work. Cut a $\triangle ABC$ from paper, as shown in Fig. 94. Put the paper triangle on a sheet of paper and draw a pencil line along the sides BC and AB. Connect the end points C and A and call the new triangle thus formed $A'B'C'$. The $\triangle A'B'C'$ was formed by making two of its sides, $B'C'$ and $A'B'$, and the $\angle B'$ included between $B'C'$ and $A'B'$, equal respectively to the sides BC and AB and the $\angle B$ included between BC and AB, of the paper triangle. Can the two $\triangle$s ABC and $A'B'C'$ be made to coincide throughout? If so, how?

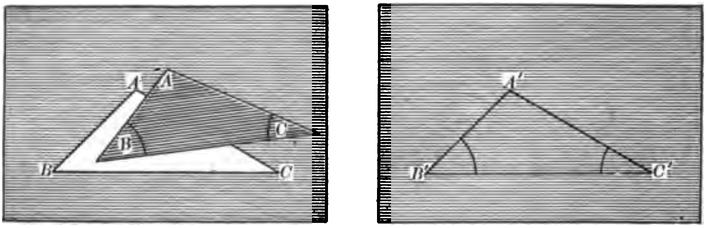


FIG. 94

98. Need for the method of proof by superposition. The method used in the preceding article for finding out whether two triangles are congruent is not accurate, it is inconvenient, and it is often impossible. In cutting out one triangle and trying to apply it to another triangle one is apt to make an error through failure to observe carefully some important part of the process used. It is clear that the method of superposition is not only more convenient but also more accurate, because it depends upon a process of reasoning alone. We need to know only the necessary equal parts in order to proceed, and these parts are usually given.

99. ***Postulates of motion.*** Superposition is really an imaginary operation, as the student will no doubt observe, and in order to superpose one geometric magnitude upon another,

Leibniz (1646–1716) was one of the greatest philosophers and mathematicians of the seventeenth century. He was a German by birth, but traveled extensively and became acquainted with the leaders in the field of mathematics all over Europe. He was one of the inventors of the calculus, a subject of great value in connection with the measurement of lines, surfaces, and solids. He and Newton had the most to do with perfecting this subject, although Newton was the first to discover the essential parts of the theory

we have to take for granted (assume) three postulates of motion, as follows:

1. *A geometric figure may be moved about in space with no other change except that of position, in such a way that any one of its points may be made to coincide with any chosen point in space.*

Thus, in Fig. 95 we may imagine the circle of radius r and center O to slide along in its plane so that its center moves along a straight line until it coincides with O', the center of a second circle with a radius given equal to r. Then the circles must coincide, and we say that they are congruent. Thus, we may prove that if the radii of two circles are equal, the circles are congruent.

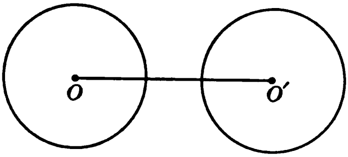


FIG. 95

2. *A geometric magnitude may also be moved about in space while one of its points remains fixed.*

Such a movement is called *rotation about a center.* This is illustrated by our definition of circle (see Art. 31), where a point P on a rotating line OP traces a path (the circle) as the line rotates about the center.

We may, of course, have occasion to slide and to rotate a geometric figure at the same time. For example, suppose we wish to superpose a line segment AB upon a line segment $A'B'$ not in the same plane, as shown in Fig. 96. To show that AB is identical with $A'B'$, let us imagine AB to slide so that A moves along the

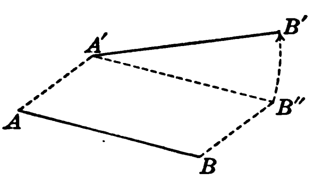


FIG. 96

straight line AA' until A coincides with A', AB remaining all the while parallel to its original position until it reaches the position $A'B''$. Then imagine $A'B''$ to rotate about A' as a center until B'' coincides with B'. Then AB and $A'B'$ coincide throughout and are therefore congruent. In the earlier stages of our mathematical work we have continually taken this second postulate for granted.

3. *A geometric figure may be moved about in space while two of its points remain fixed.*

Since two points determine a straight line, this gives us a rotation about an axis.

100. Symmetry. The third postulate in Art. 99 furnishes us with very many interesting examples of a relation called *symmetry.*

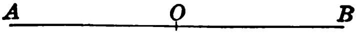


FIG. 97

101. Symmetric points. Two points are said to be *symmetric with respect to a third point* if the latter bisects the straight line connecting the two given points. The bisecting point is called the *center of symmetry.* Thus, in Fig. 97 A and B are symmetric with respect to O as a center.

102. Points symmetric with respect to a straight line. Two points are said to be *symmetric with respect to a straight line* if the straight line is the perpendicular bisector of the line connecting the two given points. The bisecting line is called the *axis of symmetry.* Thus, in Fig. 98 C and D are symmetric with respect to AB as an axis.

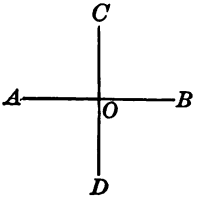


FIG. 98

103. Figures symmetric with respect to a center or an axis. A figure is *symmetric with respect to a center* when every straight line drawn through the center cuts the figure in two points which are symmetric with respect to that center. Thus, in Fig. 99 the circle is symmetric with respect to the center.

A figure is *symmetric with respect to an axis* when every straight line perpendicular to the axis cuts the figure in two points which are symmetric with respect to that axis (see Fig. 100). Axial symmetry is a very common and important property, as you will see by considering the anatomical structure of the human body, that of animals or of insects, the structure of leaves, and so on. Much use is also made of axial symmetry in designing, pattern-making, architecture, and the like.

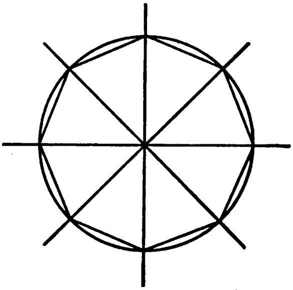


Fig. 99

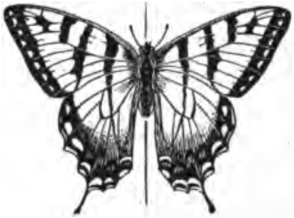


Fig. 100

104. Symmetric figures. Two figures are said to be *symmetric with respect to a center or an axis* if each figure contain the points which are symmetric to the points of the other. Thus, in Fig. 101 the $\triangle APB$ and $A'B'P'$ are symmetric with respect to CD.

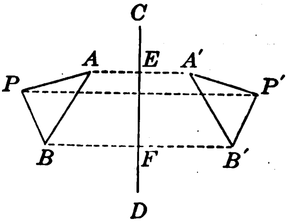


Fig. 101

From what has been said it is clear that two symmetric figures may be made to coincide. Thus, if they are symmetric with respect to a point, one can be brought into coincidence with the other by moving it around in the plane. If the two figures are symmetric with respect to an axis, one can be brought into coincidence with the other by folding the plane along the axis of symmetry.

EXERCISES

1. Fold a sheet of paper and run a pin or the point of your compasses through it. Unfold the paper and explain how this illustrates the principle in Art. 102.

2. Fold a sheet of paper and prick two holes through it. Unfold the paper, as shown in Fig. 102, and you will find two pairs of holes. Mark these, as shown, A, A'; B, B'. Show how this illustrates the principle of Art. 103.

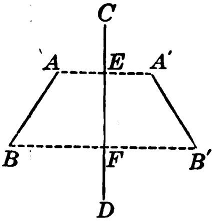


FIG. 102

3. Fold a sheet of paper and this time make three holes in it. Then unfold the paper and you will find three pairs of points, A, A'; B, B'; C, C' (Fig. 103). Join the points as shown. How does this illustrate the principle of Art. 104? How can the $\triangle ABC$ and $A'B'C'$ be made to coincide throughout?

4. Draw a figure in ink on paper and fold the paper before the ink dries. What kind of figure is obtained? How is this principle applied in printing?

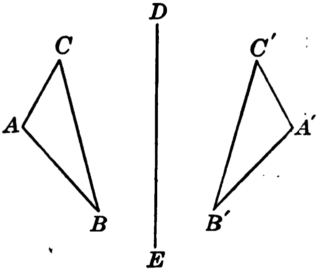


FIG. 103

5. Draw a plane figure so that it is symmetric with respect to a straight line as an axis and show that lines joining any point on the axis to two corresponding points are corresponding lines.

SUGGESTION. Explain how, if the figure is rotated about the straight line, the lines joining the two points are interchanged.

6. Fold a sheet of paper and cut out any figure you please, beginning and ending, of course, with the crease, as in Fig. 104. What kind of a figure is formed? Explain your answer. How does the ordinary tailor or dressmaker use the principle illustrated here?

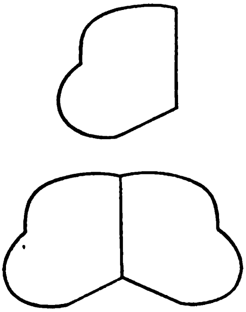


FIG. 104

105. Proposition. We have seen in Art. 69 that a theorem is the statement of a geometric relation the truth of which must be proved. We have also learned that a problem is a geometric construction to be done so as to satisfy certain given conditions. A statement of a theorem to be proved or a problem to be done is called a *proposition.*

106. Corollary. A statement the truth of which can readily be established by reference to the proof of a proposition immediately preceding is called a *corollary.*

107. Fundamental tests of congruence. Although, as we have pointed out in Art. 96, triangles must be capable of being made to coincide throughout in order to be congruent, we shall now proceed to develop certain fundamental tests of congruence for triangles in which the general conditions for congruence will be set forth. As a great deal of use is made of these theorems in future work, the student should make sure that he understands them thoroughly. This does not mean that he should try to memorize what he reads. In fact, that is what he should not do, but he should try to satisfy himself that in each case he understands clearly each statement. Moreover, he must understand the steps that are taken to arrive at the necessary conclusion so that he can make it clear to others.

108. ***Theorem.*** *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.*

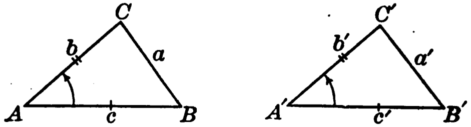


FIG. 105

Given the triangles ABC and $A'B'C'$, with b equal to b', c equal to c', and angle A equal to angle A'.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Imagine $\triangle ABC$ placed on $\triangle A'B'C'$ so that c will coincide with its equal c'.[1]	1. Any geometric figure may be moved about in space from one position to another without changing its size or shape.
2. Then, since $\angle A = \angle A'$, $\angle A$ will coincide with $\angle A'$, and b will coincide with b'.	2. Because b is given equal to b'.
3. Therefore a must coincide with a'.	3. Because only one straight line can be drawn between two points.
4. Then $\triangle ABC$ and $A'B'C'$ coincide throughout and are therefore congruent.	4. By definition of congruent triangles.

109. ***Corollary.*** *If the two sides including the right angle of one right triangle are equal respectively to the two sides including the right angle of another right triangle, the two triangles are congruent.*

---

[1] This may be done by imagining $\triangle ABC$ to slide along in the plane of $\triangle A'B'C'$ or to be picked up and superposed on $\triangle A'B'C'$. In superposition it is usually better to try to place one line so that it will coincide with another line or one angle so that it will coincide with another angle.

SUGGESTION. Since the triangles are right triangles, we know that the two right angles are equal, and the triangles are then congruent by the proposition of Art. 108. The student should give the complete statement of what is given, of what is to be proved, and of the proof.

110. Homologous parts. When two triangles are made to coincide, any selected part of one should coincide with a definite part of the other. Such parts are called *homologous* (corresponding) *parts*, and these parts may be either points, lines, or angles.

Before trying to make the homologous parts of figures coincide, it is often a good plan to designate the homologous parts by some system of marks. For example, the triangles shown in Fig. 105 are marked as follows: the equal sides AB and $A'B'$ are each marked with one short cross line, the equal sides AC and $A'C'$ are each marked with two short cross lines, and the equal angles A and A' are each marked with curved arrows.

If the figure is complicated, it is often advisable to use colored crayon to distinguish the homologous parts; but in ordinary cases the device just mentioned will be sufficient.

The student should observe that in congruent triangles homologous sides lie opposite homologous angles, that homologous angles lie opposite homologous sides, and that in all congruent figures the homologous parts are equal.

EXERCISES

1. Name all of the homologous parts of the congruent triangles in Fig. 105.

2. Could two sides of one triangle be equal respectively to the two homologous sides of another triangle and still the triangles not be congruent? Explain your answer.

3. In a given $\triangle ABC$, if $\angle A = 65°$, $\angle B = 40°$, $\angle C = 75°$, $a = 50$, $b = 35.46$, and $c = 53.29$, find the homologous parts of a $\triangle A'B'C'$ which is congruent to the $\triangle ABC$.

111. Conventional method of proving a proposition. The formal demonstration of a proposition consists of three main parts; namely, the *given part* (sometimes called the hypothesis), *what is to be proved* (called the conclusion), and the *proof.*

In writing out the proof the different statements of the argument may be numbered and placed in a column on the left side of the page, and the reason for each statement may be correctly numbered to agree with the statement and placed exactly opposite in a column on the right side of the page. Each statement must be based upon one of the following:

1. The given facts.
2. A definition previously learned.
3. An axiom or postulate already known.
4. A proposition which has been previously proved.
5. Once in a while we base a statement upon what we call *intuition,* that is, something which we accept as true without formal proof; for example, the fact that vertical angles are equal.

It is important for the student to remember that the last statement in the proof is always the statement of what is to be proved.

112. Method of proof by congruent triangles. The theorem of Art. 108 now suggests that in proving triangles congruent we need not proceed by the method of superposition each time. Thus, it is not now necessary to know that all parts of one triangle are equal to the homologous parts of the other; it is sufficient to show that the two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other. We can then quote the theorem of Art. 108 as a reason for congruence.

EXERCISES

Prove the following exercises by using the theorem of Art. 108:

1. *If two straight lines bisect each other, and their extremities are joined by straight lines, two congruent triangles are formed* (see Fig. 106).

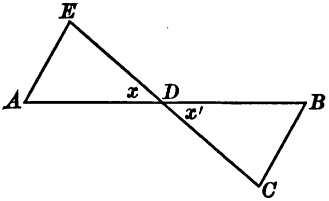


Fig. 106

Given AD equal to DB, ED equal to DC, and AE and BC drawn.

To prove that $\triangle ADE \cong \triangle DCB$.

Proof

STATEMENTS	REASONS
1. In the $\triangle$s ADE and DCB, $AD = DB$, and $ED = DC$.	1. Given.
2. $\angle x = \angle x'$.	2. When two straight lines intersect, the vertical angles are equal.
3. $\therefore \triangle ADE \cong \triangle DCB$.	3. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.

2. In Fig. 107 $\angle x = \angle x'$ and $AB = AD$. Prove that $\triangle ABC \cong \triangle ACD$.

SUGGESTION. AC is common to the $\triangle ABC$ and ACD; that is, $AC \equiv AC$ (read "AC is identical with AC").

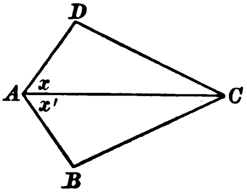


FIG. 107

3. In Fig. 108 CD is a median and $\angle x = \angle x'$. Prove that $\triangle ADC \cong \triangle CDB$.

4. In Fig. 109 $AB = CD$, and $\angle y = \angle y'$. Prove that $\triangle ABD \cong \triangle BCD$.

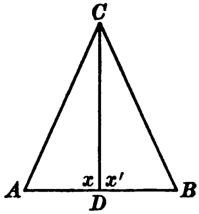


FIG. 108

5. In Fig. 110 $\angle x = \angle x'$ and $AC = CB$. Prove that $\triangle ADC \cong \triangle DBC$.

6. State the result of Ex. 5 as a theorem.

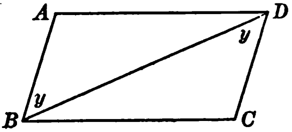


FIG. 109

113. Proving lines or angles equal. In order to prove that certain lines or certain angles are equal, try to show that they are homologous parts of congruent triangles. There are other methods of proving that lines or angles are equal, but the method of congruent triangles is a common method.

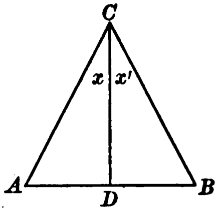


FIG. 110

In order to indicate clearly the homologous parts of congruent triangles, it is a good plan to letter the triangles with the letters A, B, C and A', B', C', as suggested in Art. 96, and also to mark the homologous angles with the letters x and x', the homologous sides with the letters a, a'; b, b'; and so on.

EXERCISES

1. In Fig. 111 $AP = PB$, $\angle x = \angle x'$, and $CA = BD$. Prove that $CP = PD$.

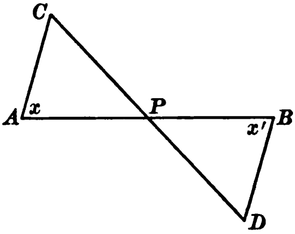


FIG. 111

Given AP equal to PB, angle x equal to angle x', and CA equal to BD.

To prove that $CP = PD$.

Proof

STATEMENTS	REASONS
1. In the $\triangle APC$ and PDB, $AP = PB$, $\angle x = \angle x'$, and $CA = BD$.	1. Given.
2. Then $\triangle APC \cong \triangle PDB$.	2. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
3. $\therefore CP = PD$.	3. Homologous parts of congruent triangles are equal.

NOTE. It is easy to see that CP and PD are homologous sides, since they lie opposite the equal $\angle$ x and x' respectively.

2. If two lines AB and CD bisect each other at P, prove that $AD = CB$.

3. In Fig. 112 $\angle x = \angle x'$, and $AC = CB$. Prove that $AD = DB$.

4. State the result of Ex. 3 as a theorem.

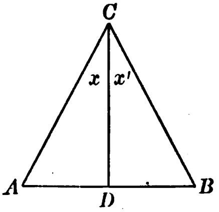


FIG. 112

5. If oblique line segments are drawn from a point in a perpendicular to a line so as to cut off equal distances on the line from the foot of the perpendicular, prove that the segments are equal and make equal angles with the perpendicular.

6. In Fig. 113 $BD = AC$, and $\angle DBA = \angle CAB$. Prove that $BC = AD$.

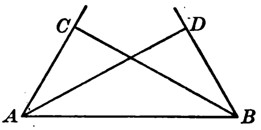


FIG. 113

114. Application of the first congruence proposition to the problem of measuring an inaccessible distance. AB (Fig. 114) represents the distance across a swamp which it is necessary to measure. To determine this distance we may place a stake at any convenient point O and by sighting from A locate a point A' in the prolongation of AO such that $A'O = OA$. In like manner, we may prolong BO until $OB' = BO$. Then, in order to find the length of AB we need only to measure the distance $B'A'$.

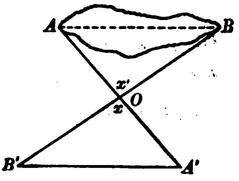


FIG. 114

Proof

STATEMENTS	REASONS
1. In the $\triangle s\ AOB$ and $A'OB'$, $AO = OA'$ and $BO = OB'$.	1. By construction.
2. $\angle x = \angle x'$.	2. If two straight lines intersect, the vertical angles are equal.
3. $\therefore \triangle AOB \cong \triangle A'OB'$.	3. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
4. $\therefore AB = A'B'$.	4. Homologous parts of congruent triangles are equal.

115. Construction lines. Often the lines and angles whose equality is to be proved are not parts of triangles at all. If an additional line or two is drawn in the figure, properties will be revealed which would not otherwise be discovered, and which will lead to a proof through the recognition of some connection with a previous proposition. Such lines are called *construction lines* and are often dotted or colored to indicate the fact that they are not part of the original figure.

No general rule can be given for drawing construction lines. It will be necessary sometimes for the student to experiment with several such lines before a method of proof can be discovered. The student will find it desirable, especially at first, to draw all construction lines carefully and accurately, employing ruler and compasses so as to make the accuracy of the construction reveal, if possible, the method of proof. After the use of these instruments is thoroughly understood the student should practice drawing rapidly but neatly, freehand or with the aid of the ruler, so that he may learn to discover proofs quickly.

We shall now prove a theorem where it is necessary to employ a construction line in order to complete the proof.

In the case of this particular theorem the student should actually do the construction referred to so as to familiarize himself with the necessary construction work.

NOTE. In the case of construction points or lines it is always assumed that the construction lines are in the same plane with the figure to which they are added.

We have in plane geometry three fundamental constructions, with ruler and compasses, which form the basis for all others; namely, (1) drawing a straight line between two points, (2) producing any straight-line segment indefinitely, and (3) drawing a circle with any point as a center and with a given radius.

116. Theorem. *If two sides of a triangle are equal, the angles opposite the equal sides are equal.*

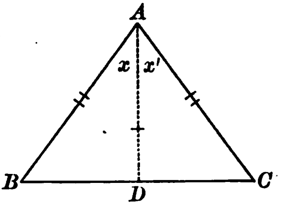


FIG. 115

Given the triangle *ABC*, with *AB* equal to *AC*.

To prove that $\angle B = \angle C$.

Proof

STATEMENTS	REASONS
1. Draw AD, the bisector of the $\angle A$, making $\angle x = \angle x'$.	1. This construction line is suggested by the discussion of Art. 115.
2. In the $\triangle$s ABD and DAC, $AB = AC$.	2. Given.
3. $\angle x = \angle x'$.	3. By construction.
4. $AD \equiv AD$.	4. Obvious.
5. $\therefore \triangle ABD \cong \triangle ADC$.	5. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
6. $\therefore \angle B = \angle C$.	6. Homologous parts of congruent triangles are equal.

117. *Corollary 1.* *The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.*

118. *Corollary 2.* *An equilateral triangle is also equiangular, and each angle of the triangle is 60°.*

EXERCISES

1. In Fig. 116 the $\triangle ABC$ is isosceles. Prove that $\angle x = \angle x'$.

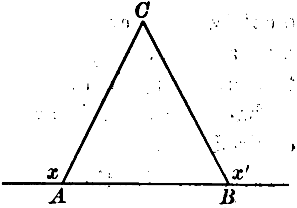


FIG. 116

2. If the perpendicular bisector of one side of a triangle passes through the vertex of the opposite angle, the triangle is isosceles.

3. All points of the perpendicular bisector of a line segment are equidistant from the ends of the line segment.

4. In an isosceles triangle the medians drawn from the ends of the base to the opposite sides are equal.

5. In Fig. 117 the $\triangle ABC$ is isosceles and CE is laid off equal to CD. Prove that $AD = BE$.

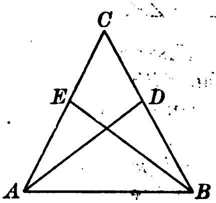


FIG. 117

6. The triangle formed by joining the midpoints of the sides of an equilateral triangle is itself an equilateral triangle.

7. The bisectors of the base angles of an isosceles triangle intersect at an angle which is equal to either exterior angle at the base of the triangle.

8. The medians of an equilateral triangle are equal.

9. The bisectors of the base angles of an isosceles triangle are equal.

10. The three bisectors of the angles of an equilateral triangle are equal.

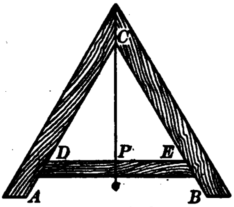


FIG. 118

11. A simple instrument for leveling, often called a *carpenter's level* or *plumb level*, consists of three sticks fastened together in the form of an isosceles triangle, as shown in Fig. 118. A plumb bob is fastened on a hook, or pivot, at C. To use the instrument it is held in an upright position so that the legs CA and CB rest upon the surface to be leveled. Show that if the plumb line hangs directly over P, the surface of the object upon which the feet rest is level.

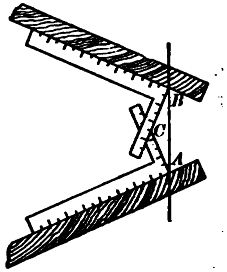


FIG. 119

12. In order to get a line AB (Fig. 119) to cut two converging pieces of timber at equal angles, a carpenter would place two steel squares against the timbers, as shown, so that $AC = CB$. Prove that AB is the required line.

13. In Fig. 120 the equal sides CA and CB of the isosceles $\triangle ABC$ are produced to D and E respectively, making $AD = BE$. Prove the theorem of Art. 116 by using this figure.

SUGGESTION. Show first that $\triangle DBC \cong \triangle AEC$, then that $\triangle ADB \cong \triangle AEB$, from which show that $\angle DAB = \angle ABE$ and $\angle x = \angle x'$.

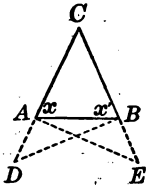


FIG. 120

HISTORICAL NOTE. The preceding exercise was the fifth proposition of Euclid's geometry and, because it was so difficult for so many students to solve, came to be known as the "pons asinorum," or *bridge of asses.* The idea of a bridge resulted from the fact that the figure resembled somewhat the framework of a bridge.

119. Directions for proving propositions. There is no one particular method by means of which all theorems or problems may be proved or solved. It is possible, however, to give certain general directions and methods which may be helpful. A knowledge of these directions will keep the student from groping about blindly and aimlessly for a proof. We shall discuss some of these methods more in detail a little later, but for the present the student will no doubt find the following general plan of attack very helpful:

1. Read the proposition carefully before you begin. Be sure that you get clearly in mind what is said and keep it in mind as you work.

2. If the proposition demands a figure, draw the figure carefully, constructing it, if possible, according to the given conditions. Thus, if a line is given equal to another line, draw it approximately equal, since this will often help you to see the relations leading to a proof. Make the figure general; that is, if the proposition refers to a triangle, draw a general triangle, never a right, isosceles, or equilateral triangle unless it is required. This will save you the error of proving the proposition for a special case only. Letter the figure properly and mark the parts which are given equal.

3. Decide upon what is given and what is to be proved, making all statements refer specifically to the figure you have drawn. Thus, instead of saying "Given an isosceles $\triangle ABC$," say "Given the $\triangle ABC$, with $AB = AC$."

(a) If the proposition is stated in the "if ...., then ...." form, then the "if" part is the hypothesis, or given part, and the remaining part is the conclusion.

(b) If the proposition is not stated as given above, then the declarative sentence in its simplest form will suggest the conclusion, and the complete subject will suggest the hypothesis. Thus, in the statement "An equilateral triangle is equiangular," the hypothesis is given by the complete subject "An equilateral triangle" and the conclusion by "is equiangular."

4. Study out a plan for your proof.

(a) Look at what is given, then at what is to be proved, and try to find out what should be added to the hypothesis to establish the conclusion. Thus, if you are trying to prove two lines equal, ask yourself, "What are the methods of proving lines equal?" Then select the proposition best suited to your needs. For example, let us suppose that we want to prove that $AC = BC$.

First Question. How can I prove that $AC = BC$?

Answer. By showing that they are homologous parts of congruent triangles.

Second Question. What are the correct triangles?

Answer. The two triangles in which AC and BC are sides.

Third Question. How can I prove that these two triangles are congruent?

Answer. By the method given in Art. 108.

5. Remember that the conclusion may often be reached by drawing construction lines.

6. Avoid unnecessary repetitions in a proof, especially if it is a long one. Thus, if two lines, AB and $A'B'$, can be proved equal in the same way that two other lines, BC and $B'C'$, have previously been proved equal, just say "In like manner, $AB = A'B'$," or "$AB = A'B'$ for the same reason." The student must be certain that the situations are the same.

EXERCISES

1. What was the hypothesis in the proposition of Art. 116? What was the conclusion?

2. Pick out the hypothesis and the conclusion in each of the following statements:

(a) *If a man is dishonest, he will not long hold the respect of his fellows.*

(b) *If powder is touched with a lighted match, it will burn or explode.*

(c) *A man true to himself will be true to other men as well.*

(d) *Honesty is the best policy.*

3. Make two statements similar to (a) and (d) above, and pick out the hypothesis and conclusion in each.

4. Make a statement which you think is true, but which we have not proved, about some geometric figure. Pick out the hypothesis and conclusion in your statement.

120. *Theorem.* *If three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent.*

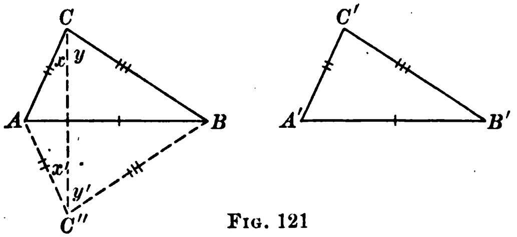


FIG. 121

Given the triangles ABC and $A'B'C'$, with AB equal to $A'B'$, AC equal to $A'C'$, and BC equal to $B'C'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Imagine $\triangle A'B'C'$ placed on $\triangle ABC$ so that $A'B'$ will coincide with its equal AB and so that C' will fall at C''.	1. Any geometric figure may be moved about in space from one position to another without changing its size or shape.
2. Draw CC'' forming $\angle$s x and y at C and $\angle$s x' and y' at C''.	2. Only one straight line can be drawn between two points.
3. In the $\triangle$s ABC and $AC''B$, $AC = AC''$.	3. Given.
4. $\therefore \angle x = \angle x'$.	4. The base angles of an isosceles triangle are equal.
5. In like manner, $CB = C''B$.	5. Given.
6. Also $\angle y = \angle y'$.	6. Same reason as 4, above.
7. $\therefore \angle x + \angle y = \angle x' + \angle y'$, or $\angle ACB = \angle AC''B$.	7. If equals are added to equals, the sums are equal.
8. $\therefore \triangle ABC \cong \triangle AC''B$.	8. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
9. $\therefore \triangle ABC \cong \triangle A'B'C'$.	9. By substitution.

EXERCISES

1. Show how the theorem of Art. 120 is an outgrowth of the proposition of Art. 108.

2. Suppose three rods to be hinged together in the form of a triangle. Could the sides be moved so as to change the angles? Is the figure rigid? Explain your answer.

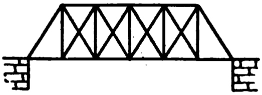


FIG. 122

3. Do four rods hinged together form a rigid figure? Give reasons for your answer.

4. How many braces (diagonals) would it take to make a triangle rigid? a quadrilateral? a pentagon?

5. Is a quadrilateral determined if four sides and either diagonal are given in a fixed order? Answer the same question if four sides and one angle are given.

6. No doubt the student has noticed that the trusses of a bridge, scaffolding, roof trusses, and other similar structures consist of a network of triangles. Why is this so? See Figs. 122 and 123.

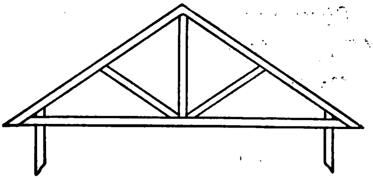


FIG. 123

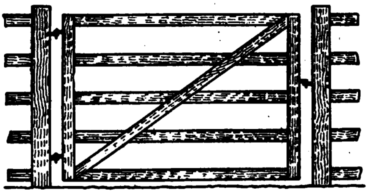


FIG. 124

7. Why is a wooden gate (Fig. 124) sufficiently braced (in theory) if a board runs diagonally across it? Why is it that in actual practice such a gate may sag?

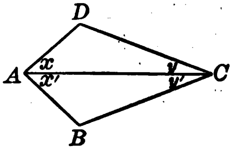


FIG. 125

8. In Fig. 125 $AB = AD$ and $BC = CD$. Prove that AC bisects the $\angle BAD$ and also bisects the $\angle BCD$.

9. Two circles whose centers are O and O' (Fig. 126) intersect at the points A and B. Prove that $\triangle OO'A \cong \triangle OBO'$.

10. A quadrilateral having two pairs of adjacent sides equal is called a *kite*. Show that AC, one of the diagonals (Fig. 127), divides a kite into two congruent triangles, and that DB, the other diagonal, divides the kite into two isosceles triangles.

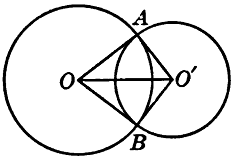


FIG. 126

11. If the opposite sides of a quadrilateral are equal, the opposite angles are equal.

12. The median to the base of an isosceles triangle bisects the vertex angle and is perpendicular to the base.

13. If the opposite sides of a quadrilateral are equal, they are parallel.

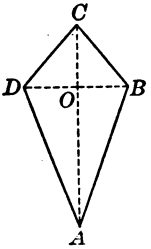


FIG. 127

14. Two triangles are congruent if two sides and a median to one of these sides are equal respectively to two sides and the homologous median of the other.

15. If the opposite sides of a quadrilateral are equal, the diagonals bisect each other.

16. If the diagonals of a quadrilateral bisect each other, the opposite sides of the quadrilateral are equal.

17. In Fig. 128 $AB = AC$ and AD bisects the $\angle BAC$. Prove that $\angle EBD = \angle ECD$.

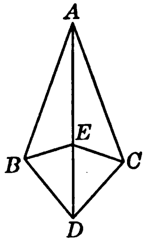


FIG. 128

SUGGESTION. In the $\triangle EBD$ and EDC the $\angle EBD$ and ECD are homologous parts. Hence, in order to prove them equal we must prove that $\triangle ABE \cong \triangle AEC$. This will give $BE = EC$, and then the rest of the proof follows.

NOTE. The last exercise illustrates the fact that to prove two lines or angles equal, it is often necessary to prove two or more pairs of triangles congruent.

121. Experimental work. Cut a $\triangle ABC$ from paper, as shown in Fig. 129. Place the paper triangle on another piece of paper and draw a pencil line along the side BC. Then draw a line a part of the way along CA and BA.

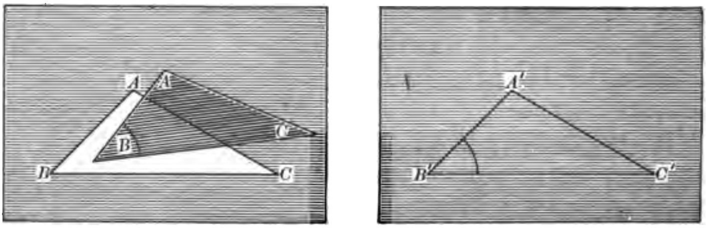


FIG. 129

Remove the cut-out triangle and produce the lines CA and BA till they meet. Call the new triangle thus formed $A'B'C'$. This new triangle was formed by making a side $B'C'$ and two adjacent ∠s B' and C' equal respectively to the side BC and the ∠s B and C. Can the two triangles be made to coincide throughout?

122. *Theorem.* *If two angles and the included side of one triangle are equal respectively to two angles and the included side of the other triangle, the triangles are congruent.*

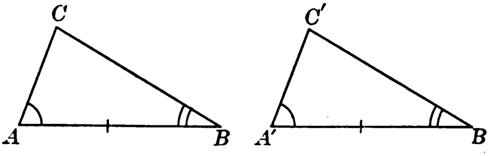


FIG. 130

Given the triangles ABC and $A'B'C'$, with angle A equal to angle A', angle B equal to angle B', and AB equal to $A'B'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Imagine $\triangle ABC$ placed on $\triangle A'B'C'$ so that AB will coincide with its equal $A'B'$ and so that A will coincide with A'.	1. Any geometric figure may be moved about in space from one position to another without changing its size or shape.
2. Then $\angle A$ will coincide with $\angle A'$, and point C will fall somewhere on the line $A'C'$ or its prolongation.	2. $\angle A$ is given equal to $\angle A'$.
3. In like manner, $\angle B$ will coincide with $\angle B'$, and point C will fall somewhere on the line $B'C'$ or its prolongation.	3. $\angle B$ is given equal to $\angle B'$.
4. Since C' falls both on $A'C'$ and $B'C'$, it must fall at their point of intersection.	4. Two straight lines cannot intersect in more than one point.
5. $\therefore \triangle ABC \cong \triangle A'B'C'$.	5. By definition of congruent triangles.

123. Corollary 1. *If two angles and any side of one triangle are equal respectively to two angles and the homologous side of another triangle, the triangles are congruent.*

124. Corollary 2. *Two right triangles are congruent if a leg and an acute angle of one are equal respectively to a leg and an acute angle of the other.*

EXERCISES

1. In $\triangle ABC$ (Fig. 131) CD bisects $\angle C$ and $CD \perp AB$. Prove that $AD = DB$.

2. The $\angle s\ A$ and B of $\triangle ABC$ (Fig. 132) are equal, and AB is bisected at D. The lines ED and DF are drawn so that $\angle x = \angle x'$. Prove that $\triangle ADE \cong \triangle DBF$.

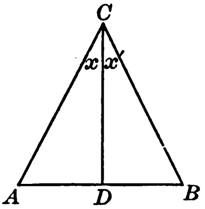


FIG. 131

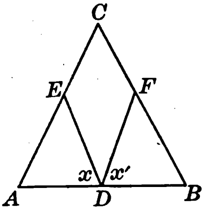


FIG. 132

3. In Fig. 133 $ABCD$ is a square. M is the midpoint of AB, and the lines EM and MF are drawn so that $\angle x = \angle x'$. Prove that $EA = FB$.

4. If a diagonal of a quadrilateral (Fig. 134) bisects the angles whose vertices it joins, the two triangles into which the diagonal divides the quadrilateral are congruent.

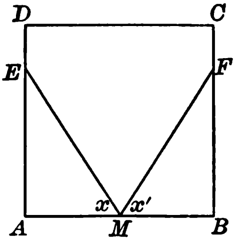


FIG. 133

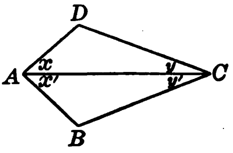


FIG. 134

5. In Fig. 135 P is the midpoint of AB. AD and BC are each perpendicular to AB. Prove that $\triangle APD \cong \triangle PCB$.

6. If a line bisects an angle of a triangle and is perpendicular to the opposite side, the triangle is isosceles.

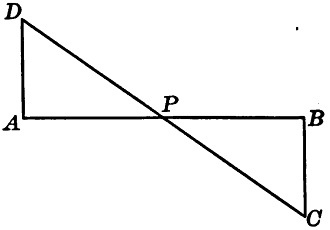


FIG. 135

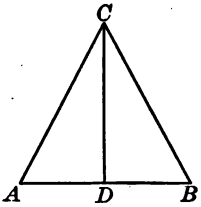


FIG. 136

7. In $\triangle ABC$ (Fig. 136) $AC = BC$ and CD bisects $\angle C$. Show by the theorem of Art. 108 that $AD = DB$.

8. Oblique lines drawn to a given line from a point on a perpendicular to the line and making equal angles with the perpendicular are equal.

9. Oblique lines drawn from a point on a perpendicular to a given line and making equal angles with the line are equal.

10. If at any point in the bisector of an angle perpendiculars to the bisector are drawn to each side of the angle, the two triangles thus formed are congruent.

11. If two angles of a triangle are equal, the bisectors of these angles are equal.

SUGGESTION. Prove that $\triangle BCD \cong \triangle BCE$ (Fig. 137).

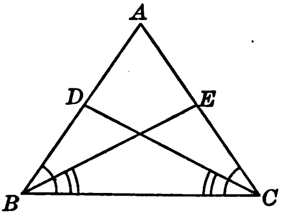


FIG. 137

12. If two triangles are congruent, the bisector of any angle of one is equal to the bisector of the homologous angle of the other.

SUGGESTION. In Fig. 138 prove that $DB = D'B'$.

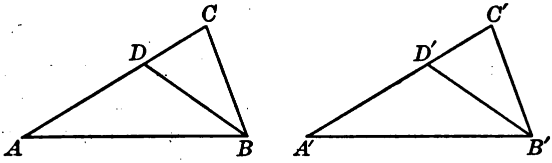


FIG. 138

13. In $\triangle ABC$ (Fig. 139) $\angle ABC = \angle CAB$, AD bisects $\angle CAB$, and BE bisects $\angle ABC$. Prove that $AD = BE$.

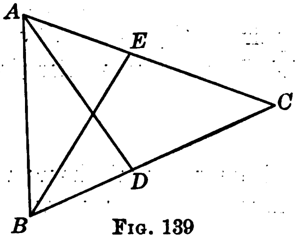


FIG. 139

125. Dual theorems; Principle of Duality. The student will observe that the propositions of Arts. 108 and 122 have a certain similarity. If the words "side" and "angle" are interchanged in the proposition of Art. 108, we have the proposition of Art. 122; if interchanged in the proposition of Art. 122, we have the proposition of Art. 108. Theorems that have this relation are called *dual theorems*, and the principle involved is called the *Principle of Duality.*

The relation can be more clearly seen if the two propositions are written in parallel columns, thus:

ART. 108. *If two* sides *and the included* angle *of one triangle are equal respectively to two* sides *and the included* angle *of another triangle, the triangles are congruent.*	ART. 122. *If two* angles *and the included* side *of one triangle are equal respectively to two* angles *and the included* side *of another triangle, the triangles are congruent.*

The student should also observe that if the small letters *a*, *b*, and *c* are interchanged for the capital letters *A*, *B*, and *C* respectively in the proof of the proposition on the left, the proof becomes that of the proposition on the right.

However, the student must not get the idea that because a theorem is true its dual theorem is necessarily true. In geometric work involving measurements the dual is often false, as we shall see. The principle is valuable, nevertheless, for it helps the student to see the relation between propositions and often gives him a clue in proving new theorems.

For many theorems in geometry dual theorems may be formed by replacing (1) "point" by "line," (2) "line" by "point," (3) "angles" of a triangle by (opposite) "sides," (4) "sides" of a triangle by (opposite) "angles."

EXERCISE

State the dual theorem of Corollary 1, Art. 123, and tell whether it is true or not. Give reasons for your answer.

126. Application of the third congruence proposition to the problem of measuring an inaccessible distance. Some boys at a summer camp desired to know the approximate distance across the narrow neck of a lake where they were swimming. They had no measuring instruments; but one of the boys stood on one side of the lake, as at *A*.

(Fig. 140) and sighted over his thumb, extended at arm's length, at a point B, just across the lake. Then, without lowering or raising his arm, he turned "about face" and located a second point C by sighting over his thumb. He then paced the distance AC to get the required distance across the neck of the lake. Was his method such that it gave him an approximately correct result? Explain fully.

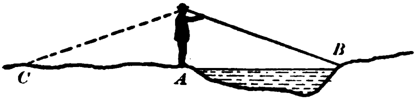


FIG. 140

EXERCISES

The propositions of Arts. 108, 120, and 122 may be applied in a practical way to many problems where the measurement of inaccessible distances is required. See if you can apply them to the following:

1. The theorem of Art. 122 is said to have been used by Thales, the founder of the earliest Greek school of mathematics (640 B.C. to about 550 B.C.), to determine the distance AV of a vessel V (Fig. 141) from the shore. This was done by measuring the $\angle$s x and y and then constructing on shore the $\triangle ACB$, congruent to the $\triangle ABV$. Show how this can be done. Find AV and BV if $BC = 4820$ ft. and $AC = 3280$ ft.

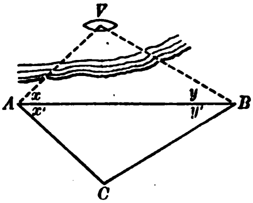


FIG. 141

2. Show that the distance AB across a river (Fig. 142) may be obtained by the following process:

(a) Lay off $BC \perp AB$ and $CE \perp BC$.

(b) Drive a stake at P, the midpoint of BC.

(c) Then determine, by sighting, a point D on CE such that A, P, and D will be in the same straight line, and measure CD.

Prove that $CD = AB$.

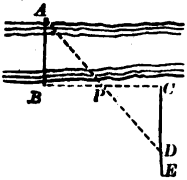


FIG. 142

3. The height of a flagpole (Fig. 143) may be found by laying off on the ground a $\triangle A'B'C'$ which will be congruent to the $\triangle ABC$. Show how this can be done.

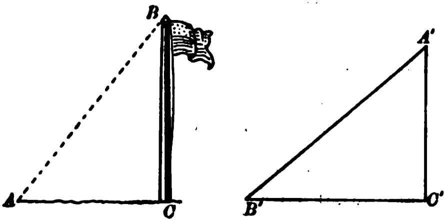


FIG. 143

4. Thales is also said to have made an instrument for determining the distance of a ship from shore (see Ex. 1). Two pointers RA and RB (Fig. 144) were fastened at R. RB was held vertically over the point C on the shore while at the same time RA was pointed at the ship at D. Then, without changing the $\angle ARB$ the instrument was revolved about RB as an axis until some point E was located in line with RA'. The distance to the point E was then measured. Show why $CD = EC$.

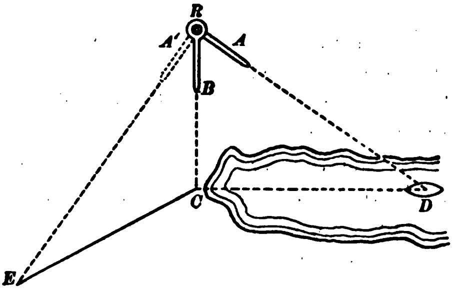


FIG. 144

5. In former days a device called a *cross staff* was used for measuring inaccessible distances. It was composed of a vertical staff VA to which was attached a movable cross bar EC that could move up and down and also revolve about P. Show how this instrument might be used to measure the distance AB (Fig. 145), giving the necessary proof to show that the results obtained by this method were correct.

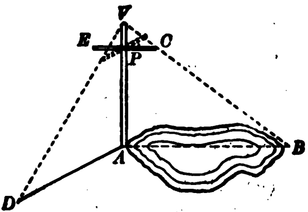


FIG. 145

127. Theorem. *If two angles of a triangle are equal, the sides opposite the equal angles are equal.*

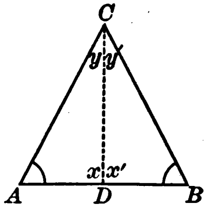


FIG. 146

Given the triangle ABC, with angle A equal to angle B.

To prove that $AC = CB$.

Proof

STATEMENTS	REASONS
1. Draw $CD \perp AB$.	1. One and only one perpendicular can be drawn from a point to a line.
2. In the $\triangle$s ADC and DBC, $\angle x = \angle x'$.	2. By construction.
3. Also, $\angle A = \angle B$.	3. Given.
4. $\therefore \angle y = \angle y'$.	4. If two angles of one triangle are equal respectively to two angles of another triangle, the third angles are equal.
5. $CD \equiv CD$.	5. Obvious.
6. $\therefore \triangle ADC \cong \triangle CDB$.	6. If two angles and the included side of one triangle are equal respectively to two angles and the included side of the other triangle, the triangles are congruent.
7. $\therefore AC = CB$.	7. Homologous parts of congruent triangles are equal.

128. Corollary. *An equiangular triangle is equilateral.*

EXERCISES

1. If the bisectors of the base angles of an isosceles triangle are produced till they meet, they form with the base another isosceles triangle.

2. If from the ends of the base of an isosceles triangle perpendiculars are drawn to the two equal sides, another isosceles triangle is formed with the base.

3. A line parallel to the base of an isosceles triangle and intersecting the two equal sides forms with the sides another isosceles triangle.

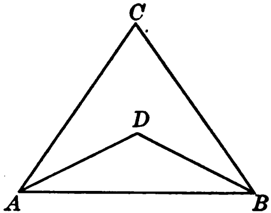


FIG. 147

4. If the bisector of an exterior angle of a triangle is parallel to one side, the triangle is isosceles.

5. If $\triangle ABC$ (Fig. 147) is isosceles, and AD and BD bisect $\angle A$ and $\angle B$ respectively, prove that $\triangle ABD$ is isosceles.

6. The inaccessible distance AC (Fig. 148) may be measured as follows: Run a line AB until a point is reached where $\angle z = \frac{1}{2} \angle x$. Then $AB = AC$. Prove it.

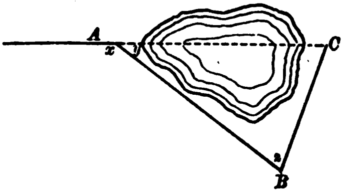


FIG. 148

7. In order to find the distance of his ship B (Fig. 149) from an object O, a mariner would proceed as follows: If AB is the direction of the ship's motion, he would observe the time when the direction of the object O makes a given angle x with AB, and again when the angle is twice as large. The distance OB will then be equal to the distance AB, which he can measure by knowing the time and rate of his vessel in sailing from A to B. Prove that this method is accurate.

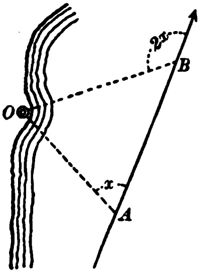


FIG. 149

129. Theorem. *If the acute angles of a right triangle are 30° and 60° respectively, the side opposite the 30-degree angle is equal to one half the hypotenuse.*

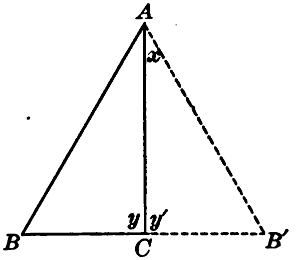


FIG. 150

Given the right triangle ABC, with angle C equal to 90°, angle A equal to 30°, and angle B equal to 60°.

To prove that $BC = \frac{1}{2} AB$.

Proof

STATEMENTS	REASONS
1. At A draw AB' making $\angle x = \angle A$.	1. An angle may be constructed equal to a given angle.
2. Extend BC to meet AB'.	2. A line may be extended.
3. Then in $\triangle ABB'$, $\angle B = 60°$.	3. Given.
4. $\angle B' = \angle B$; that is, $\angle B' = 60°$.	4. Because $\angle y' = \angle y$ by definition of a perpendicular, and $\angle x = \angle A$ by construction.
5. Also $\angle BAB' = 60°$.	5. Because $\angle BAB' = \angle A + \angle x$.
6. $\therefore \triangle ABB'$ is equilateral.	6. An equiangular triangle is equilateral.
7. $\therefore AB = BB' = AB'$.	7. $\triangle ABB'$ is equilateral.
8. But $BC = \frac{1}{2} BB'$.	8. The perpendicular to the base from the vertex of an isosceles triangle bisects the base.
9. $\therefore BC = \frac{1}{2} AB$.	9. If $BC = \frac{1}{2} BB'$, it is also one half of a line equal to BB'.

EXERCISE

State and prove the converse of the theorem above.

130. *Theorem.* *If two isosceles triangles are constructed upon the same line segment as a base, the line that joins their vertices bisects the common base at right angles.*

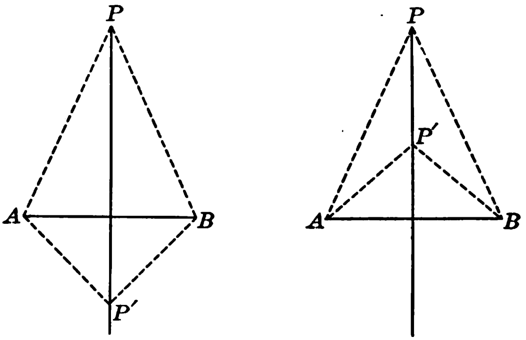


FIG. 151

Proof. The proof is left to the student (see Fig. 151).

131. *Corollary 1.* *Two points, each equidistant from the ends of a line segment, determine the perpendicular bisector of the line segment.*

132. *Corollary 2.* *The diagonals of a kite are perpendicular to each other.*

EXERCISE

If two triangles have a side of one and the altitude and median to that side equal respectively to the homologous parts of the other, the triangles are congruent.

133. Converse. The student should now observe that the theorems of Arts. 116 and 127 bear certain mutual relations. It is clear that what is given in the theorem of

Art. 116 is to be proved in the theorem of Art. 127, and that what is to be proved in the theorem of Art. 116 is given in the theorem of Art. 127. When two propositions are so related that what is given in the first is to be proved in the second, and what is to be proved in the first is given in the second, the first proposition is said to be the *converse* of the second and vice versa. Thus, the theorems of Arts. 116 and 127 are *converse propositions*, and each is said to be the converse of the other.

The converse of a proposition can always be stated, *but it may not be a true proposition.* For example, "All straight angles are equal" is a proposition whose converse "All equal angles are straight angles" is not true. Therefore the converse of a proposition must always be proved if it is to be used as a separate proposition.

Outside of the field of mathematics we find many instances of statements and their converses. Here again the fact that a statement is true gives us no absolute assurance that its converse is true, as is clear from the following examples:

STATEMENT. If a man is rich, he can buy a knife. (*True.*)
CONVERSE. If a man can buy a knife, he is rich. (*False.*)

EXERCISES

1. Make a true statement whose converse is false.
2. Make a true statement whose converse is true.
3. Pick out two propositions and their converses which are given in the preceding pages of this text.

134. Congruence of polygons. The congruence of two polygons can be proved by superposition, as was done in the case of triangles. Just as a polygon of three sides has six parts (three sides and three angles), so a polygon of four sides has eight parts, one of five sides has ten parts, and, in general, one of n sides has $2n$ parts, of which n parts are sides and n are angles. It can be proved that *if two polygons of n sides have $2n - 3$ consecutive parts of one equal respectively to $2n - 3$ consecutive parts of the other, the polygons are congruent.* However, we shall not be interested in this course, so far as congruence is concerned, with any polygon of a greater number of sides than four.

135. *Theorem.* *Two right triangles are congruent if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.*

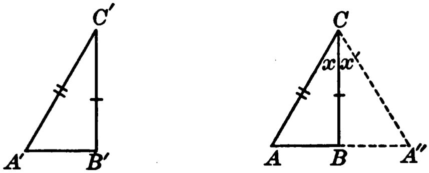


FIG. 152

Given the right triangles *ABC* and *A′B′C′*, with *AC* equal to *A′C′* and *BC* equal to *B′C′*.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Imagine $\triangle A'B'C'$ placed on $\triangle ABC$ so that $B'C'$ will coincide with its equal BC and so that A' falls at A''.	1. Any geometric figure may be moved about in space from one position to another without changing its size or shape.
2. Then, since $\angle ABC = \angle A''BC$,	2. Given.
3. $\angle ABC + \angle A''BC = 180°$.	3. The sum of two right angles is one straight angle.

Proof (*Continued*)

STATEMENTS	REASONS
4. $\therefore ABA''$ is a straight line.	4. Whenever the sum of two adjacent angles is a straight angle their exterior sides form a straight line.
5. Also $AA''C$ is a triangle.	5. By definition.
6. But $AC = A''C$.	6. Given.
7. $\therefore \angle A = \angle A''$.	7. Base angles of an isosceles triangle are equal.
8. $\therefore \angle x = \angle x'$.	8. When two angles of one triangle are equal respectively to two angles of another triangle, the third angles of the triangles are equal.
9. $\therefore \triangle ABC \cong \triangle CBA''$.	9. If two angles and the included side of one triangle are equal respectively to two angles and the included angle of another triangle, the triangles are congruent.
10. $\therefore \triangle ABC \cong \triangle A'B'C'$.	10. By substitution.

EXERCISES

1. Two right triangles are congruent if any two sides of one are equal respectively to any two homologous sides of the other.

2. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.

3. The altitudes drawn from the ends of the base of an isosceles triangle upon the equal sides are equal.

4. State and prove the converse of Ex. 3.

5. The perpendiculars drawn from the midpoint of the base to the equal sides of an isosceles triangle are equal.

6. The perpendiculars drawn from the midpoints of the equal sides of an isosceles triangle to the base are equal.

7. The perpendiculars erected to the base of an isosceles triangle at points equidistant from its midpoint and terminated by the equal sides are equal.

8. If two points are on opposite sides of a line and at the same distance from it, the line joining the two points is bisected by the given line at right angles.

9. State and prove the converse of Ex. 5.

10. Perpendiculars erected at the midpoints of the equal sides of an isosceles triangle and terminated by the base, or the prolongation of the base, are equal.

11. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

12. If two altitudes of a triangle are equal, the homologous sides upon which they are drawn are equal and the triangle is isosceles.

13. Two triangles are congruent if two sides and the altitude upon one of them in one triangle are equal respectively to two sides and the homologous altitude of the other.

14. Prove the theorem of Art. 135 when $AC = A'C'$ and $AB = A'B'$, placing the triangles as in Fig. 153.

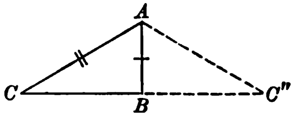


FIG. 153

15. Prove that the altitudes of an equilateral triangle are equal.

136. How to solve a construction problem. A complete solution of a construction problem consists of the following three distinct steps:

1. Making the construction with ruler and compasses.
2. Proving the construction correct; that is, showing that the figure constructed fulfills the required conditions.
3. Discussing all the possibilities; that is, stating the conditions under which there is no solution, one solution, more than one solution, and how many solutions.

137. Proofs of constructions. Up to this point we have learned how to make many fundamental constructions, but we have not been required to prove that these constructions were correct. From now on, when a construction is made, the student should prove that the construction is correct. The fundamental congruence theorems which we have been studying will be very helpful in giving this proof.

138. ***Construction problem.*** *At a given point on a given line to construct an angle equal to a given angle.*

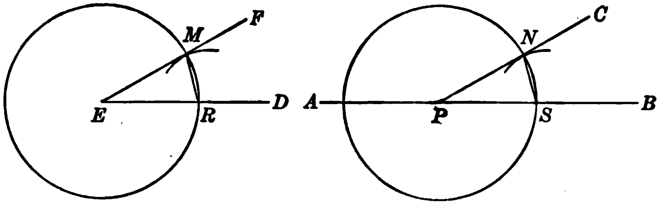


FIG. 154. CONSTRUCTING AN ANGLE EQUAL TO A GIVEN ANGLE

Given the angle DEF, the line AB, and a point P on it.

Required *to construct at P an angle equal to the $\angle DEF$.*

Construction. See Art. 41.

Proof

STATEMENTS	REASONS
1. Draw MR and NS.	1. One and only one straight line can be drawn between two points.
2. In the $\triangle$s MER and NPS, $EM = PN$, $ER = PS$, and $MR = NS$.	2. By construction.
3. $\therefore \triangle MER \cong \triangle NPS$.	3. If three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent.
4. $\therefore \angle BPC = \angle DEF$.	4. Homologous parts of congruent triangles are equal.

139. The synthetic method of proof. The *synthetic method of proof* consists in the putting together of certain known truths in order to establish a new truth. It is the method which we have used the most in the proofs of the preceding propositions, and is the one which we ordinarily use in writing out a finished proof, because in such form the proof is more elegant and concise.

The synthetic method of proof, however, is open to the objection that it often does not show why the particular order and combination of truths used are chosen in preference to other possible ones. In other words, it is often not clear to the student how the author discovered the proof. The student, therefore, gets the pernicious habit of memorizing the proof when he should be able to discover it himself. For example, let us take the proof of the preceding problem. The truth of each step is not hard to understand, but the student very often experiences difficulty in seeing in advance just why the particular truths lead to the desired conclusion. The danger, therefore, in using this method too much is that the student is apt to lose his interest and does not see the chain of reasoning by means of which the proof was really discovered. When this occurs, the memorizing of proofs results, and the very purpose of geometric proofs is defeated. The student must learn the synthetic method as a method of writing out a proof in final form, but he must also learn how to discover a proof.

140. The analytic method of proof. For the student who will work in the attitude of a discoverer, we wish to recommend the *analytic method of proof.* This method consists in separating the proposition to be proved into its proper parts, proving each of these, and thus establishing the entire proposition. The student should use this method of proof wherever the synthetic method is not readily adaptable to the given proposition. He should practice the method as much as possible even though he may finally set down his proof in synthetic form, for unless he masters this method of proof he will never be able to discover proofs readily.

In the following problem we shall use the analytic method of proof in order to show the student how it is used. We shall then give various exercises which will require the use of the method.

141. Construction problem. *To bisect a given line segment.*

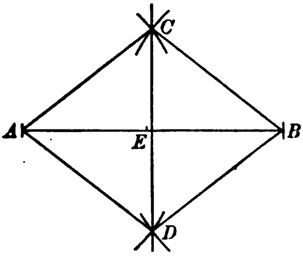


FIG. 155

Given the line segment AB.

Required *to bisect AB.*

Construction. See Art. 45.

Analysis. The problem is to prove that $AE = EB$, which we can do if we can show that the lines are homologous sides of congruent triangles. There are two pairs of triangles in which AE and EB are homologous sides; namely, $\triangle$s ACE and CEB, $\triangle$s AED and EDB. Hence we draw AC, CB, AD, and DB and then see if we can prove that $\triangle AEC \cong \triangle CED$. We have three fundamental congruence theorems which we may use, and we proceed to see which one applies here. We know that $AC = CB$ by construction, that $CE \equiv CE$, and we see that the triangles are congruent if we can prove that $\angle ACE = \angle ECB$.

In order to prove these angles equal, it is obvious that we cannot use $\triangle$s AEC and EBC, and hence we try $\triangle$s ADC and DBC. We know that $AC = CB$ and $AD = DB$ by construction, and that $CD \equiv CD$. Therefore the $\triangle ADC$ and DBC are congruent, because three sides of one are equal respectively to three sides of the other. We see now how to prove the theorem, and we may write the proof in the following synthetic form:

Proof

STATEMENTS	REASONS
1. Draw AC, CB, BD, and DA.	1. One and only one straight line can be drawn between two points.
2. In the $\triangle ADC$ and DBC, $AC = CB$, and $AD = DB$.	2. By construction.
3. $CD \equiv CD$.	3. Obvious.
4. $\therefore \triangle ADC \cong \triangle DBC$.	4. If three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent.
5. $\therefore \angle ACE = \angle ECB$.	5. Homologous parts of congruent triangles are equal.
6. But $AC = CB$, and $CE \equiv CE$.	6. By construction.
7. $\therefore \triangle AEC \cong \triangle CEB$.	7. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
8. $\therefore AE = EB$.	8. Homologous parts of congruent triangles are equal.
9. $\therefore AB$ is bisected.	9. By definition of a bisector.

EXERCISES

1. At a point P on a given line AB erect a perpendicular and prove that the construction is correct.

SUGGESTION. Connect E with C and D in Fig. 156 and then use the analytic method of proof.

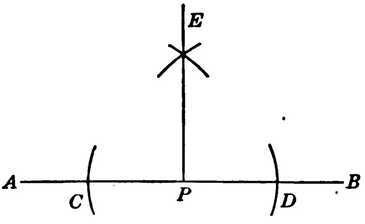


FIG. 156

2. Draw a perpendicular to a given line AB from a given point P outside AB, and prove that your construction is correct.

SUGGESTION. Use the analytic method, as in Art. 141 and Fig. 157.

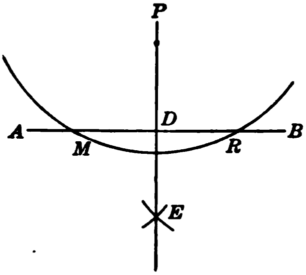


FIG. 157

3. Bisect a given $\angle ABC$ and prove that your construction is correct.

SUGGESTION. Use the analytic method. Drop perpendiculars from D to AB and BC (Fig. 158), and thus obtain two triangles.

4. Show how an angle can be bisected by use of a carpenter's steel square.

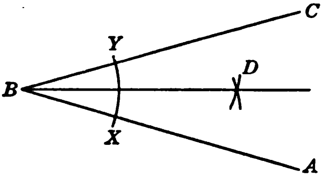


FIG. 158

5. An instrument for finding half of a given angle may be made as follows: Pivot three pieces of wood together at A, O, and C (Fig. 159) so that $AO = OC$. The pivot A slides in a groove on AC, so that as AC is shortened or lengthened, the bar OC rotates about the pivot O and thus enlarges or reduces the $\angle BOC$. Show that if the $\angle BOC$ is superposed on any given angle the $\angle OAC$ will always be equal to one half the given angle.

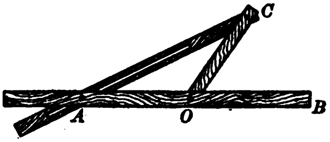


FIG. 159

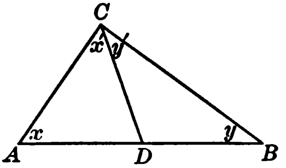


FIG. 160

6. In Fig. 160 CD bisects AB and is equal to $\frac{1}{2}$ AB. Prove that $x' + y' = 90°$.

142. Geometrical fallacies. The importance of being careful, both in solving construction problems and in working out a geometric proof, may be illustrated by a seemingly valid proof which, however, upon careful examination is found to contain false reasoning. Such propositions are called *geometrical fallacies.* For example, let us set ourselves the task of proving that *all triangles are isosceles.* We may proceed as follows*:

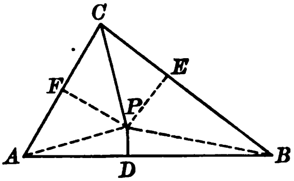


FIG. 161

Given the triangle *ABC*.

To prove that $AC = BC$.

Proof

STATEMENTS	REASONS
1. Let DP be the perpendicular bisector of AB, CP the bisector of $\angle C$, and let the two bisectors meet at P.	1. Two lines which are not parallel must meet if sufficiently produced.
2. Draw PA and PB.	2. One and only one line can be drawn between two points.
3. Then $AP = PB$.	3. Any point on the perpendicular bisector of a line segment is equidistant from the end points of that line segment.
4. Draw $PF \perp AC$ and $PE \perp BC$.	4. From a point outside a given line one and only one perpendicular can be drawn to the line.

Proof (*Continued*)

STATEMENTS	REASONS
5. $\angle FCP = \angle PCE$.	5. By construction.
6. $\angle CPF = \angle CPE$.	6. Complements of equal angles are equal.
7. $CP \equiv CP$.	7. Obvious.
8. $\therefore \triangle CPF \cong \triangle CPE$.	8. Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.
9. $\therefore PF = PE$ and $CF = CE$.	9. Homologous parts of congruent triangles are equal.
10. $\therefore \triangle APF \cong \triangle PBE$.	10. Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.
11. $\therefore AF = BE$.	11. Homologous parts of congruent triangles are equal.
12. Since $CF = CE$,	12. Proved in 9.
13. Then $CA = CB$.	13. Equals added to equals give equals.
14. $\therefore \triangle ABC$ is isosceles.	14. By definition.

EXERCISES

1. Make a careful construction of Fig. 161, according to the directions given, and try to explain the fallacy.

2. We learn in science that if a ray of light x (Fig. 162) strikes a plane mirror MN at B, it is reflected in such a way that the *angle of incidence* ($\angle ABD$) *is equal to the angle of reflection* ($\angle DBC$). Study the figure and explain why this is true.

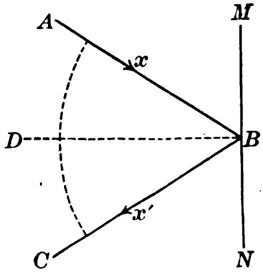


FIG. 162

3. If a ray of light strikes the mirror MN (Fig. 162) perpendicularly, how will it be reflected? Why?

4. If a ray of light strikes first one mirror and then another in such a way that the path of the ray forms an equiangular triangle, construct the figure and find the angle between the two mirrors.

SUGGESTION. Assume that the two mirrors are perpendicular to the plane of the triangle formed by the path of the ray.

5. If a ray of light strikes first one mirror and then another in such a way that it is reflected from the second mirror along a line which is parallel to its path before it struck the first mirror, find (1) by construction and (2) by computation the angle between the mirrors.

6. If the median of a triangle is perpendicular to the base, the triangle is isosceles.

7. Homologous medians of congruent triangles are equal.

8. Two triangles are congruent if they have a side, an adjacent angle, and the bisector of that angle of one respectively equal to the homologous parts of the other.

9. If the altitude of a triangle bisects the vertex angle, the triangle is isosceles.

10. The altitude to the base of an isosceles triangle bisects the vertex angle.

11. Homologous altitudes of congruent triangles are equal.

CHAPTER III

POLYGONS

143. Inductive method. The student may often discover important mathematical relations by studying a number of special cases which exemplify laws, principles, or facts which he will need to use in his later work. This method of discovery is called the *inductive method* and consists in studying certain special cases in order to discover the general law, definition, or principle which is true for all cases of which the chosen cases are special ones. The method can probably be best understood by studying the method of solving the following problem, where the inductive method is used.

144. ***Problem.*** *To find the sum of the interior angles of any n-gon.*

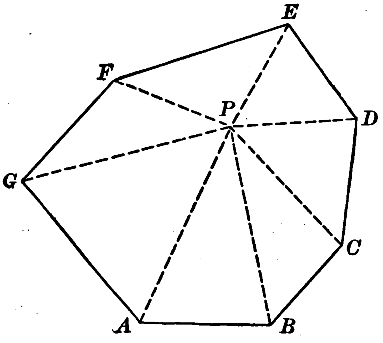


FIG. 163

Given an *n*-gon *ABCDEFG* · · ·.

Required *to find the sum of the interior angles of the n-gon.*

Inductive reasoning. In order to solve this problem inductively we must study several different n-gons and thus see if we can discover the common fact concerning the sum of the interior angles in each case. Some of the well-known n-gons are the triangle, the quadrilateral, the pentagon, the hexagon, the heptagon, the octagon, the decagon, and the dodecagon; that is, polygons of 3, 4, 5, 6, 7, 8, 10, and 12 sides respectively.

Using the special cases of the n-gon given above, make a table similar to the one given below showing the sum of the interior angles in each polygon. For example, the triangle has three sides, and the sum of its interior angles we know is one straight angle, or two less than the number of sides; the quadrilateral has four sides, and we have proved (Art. 83, Ex. 3) that the sum of its interior angles is two straight angles, again two less than the number of sides.

Work out similar facts for the other special polygons and tabulate the results, as shown in the following table:

TYPE OF POLYGON	NUMBER OF SIDES	NUMBER OF TRIANGLES	SUM OF THE INTERIOR ANGLES OF EACH TRIANGLE	SUM OF THE INTERIOR ANGLES OF THE POLYGON
Triangle	3	1, or $3 - 2$	1 st. ∠	$(3 - 2)$ st. ∠s
Quadrilateral . . .	4	2, or $4 - 2$	1 st. ∠	$(4 - 2)$ st. ∠s
Pentagon	5	3, or $5 - 2$	1 st. ∠	$(5 - 2)$ st. ∠s
Hexagon				
Heptagon				
Octagon				
Decagon				

By examining the complete table carefully it is easy to see that the sum of the interior angles of each polygon is always two straight angles less than the number of sides in the polygon. This leads us to have faith in the general truth of the preceding fact, which we may state as follows: "The sum of the interior angles of any polygon is $(n - 2)$ straight angles, where n is the number of sides."

However, we cannot always be sure that a thing is always true simply because it is true a few or even many times. Therefore we must prove the theorem.

Proof

STATEMENTS	REASONS
1. Take any point P inside the given polygon and connect it with each vertex of the polygon.	1. One and only one straight line can be drawn between two points.
2. Then there are as many triangles as the polygon has sides; that is, there are n triangles in a polygon of n sides.	2. Obvious.
3. The sum of the interior angles of each triangle of the polygon is one straight angle.	3. The sum of the interior angles of a triangle is a straight angle.
4. ∴ the sum of all the interior angles of the n triangles is n straight angles.	4. Because it is n times one straight angle.
5. But the sum of all the angles around the point P is two straight angles.	5. The sum of all the angles about a point in a plane is two straight angles.
6. ∴ the sum of the interior angles of an n-gon is $(n-2)$ straight angles.	6. By subtraction.

145. *Corollary 1.* *The sum of the interior angles of an n-gon is* $(2n-4)$ *right angles.*

146. *Corollary 2.* *Each angle of a regular polygon of n sides is equal to* $\frac{n-2}{n}$ *straight angles or* $\frac{2(n-2)}{n}$ *right angles.*

EXERCISES

1. See if you can prove the problem in Art. 144 by drawing diagonals from A to each of the nonadjacent vertices in Fig. 164.

2. Using the formula obtained in Art. 145, find the sum of the interior angles of a quadrilateral; of a pentagon; of a hexagon; of an octagon; of a decagon; of a polygon of thirty sides.

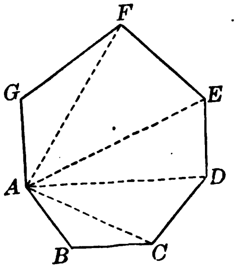


FIG. 164

3. How many degrees are there in each angle of a regular pentagon? of a regular hexagon? of a regular decagon?

4. How many sides are there in a regular polygon the sum of whose angles is eight straight angles?

5. How many sides are there in a regular polygon each of whose angles is 150°?

6. The sum of the exterior angles of any polygon made by producing each of the sides in order through a vertex is two straight angles.

7. The angles of a pentagon are x, $\frac{2x}{3}$, $2x$, $3x$, and $\frac{3x}{2}$. Find the number of degrees in each angle.

8. How many degrees are there in each exterior angle of a regular pentagon? of a regular hexagon? of a regular decagon?

9. How many sides are there in a polygon the sum of whose interior angles is 360°? 720°? 1080°? twenty straight angles?

10. The bisectors of the base angles of an isosceles triangle and those of the adjacent exterior angles form a quadrilateral in which the opposite angles are supplementary.

11. In a regular polygon any central angle is equal to an exterior angle.

12. If the sides of a pentagon are extended as shown in Fig. 165 we get a figure called a *star polygon*. This particular type of star polygon is called a *pentagram* and is known historically as the badge of the school of Pythagoras. Other star polygons may be formed by producing the sides of convex polygons or by drawing chords of circles in certain ways. What is the sum of the vertex angles of a five-pointed star? of a six-pointed star? of an n-pointed star?

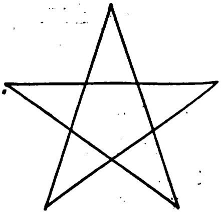


FIG. 165. PENTAGRAM

13. Find the sum of the vertex $\angle$s v, w, x, y, and z which are shown in Fig. 166.

14. Find the number of sides in a polygon if the sum of the interior angles is twice that of the exterior angles of the polygon.

15. Prove that in the regular polygon shown in Fig. 167 a central $\angle z$ is equal to an exterior $\angle x$.

16. Prove the proposition of Art. 144 by drawing lines from any point on one side of the polygon to the vertices.

17. Show that the sum of the interior angles of a polygon is a function of (depends upon) the number of sides.

18. Show that the function x^2+x+41 gives prime numbers for all values of x from 1 to 40 but that $x = 41$ does not.

19. How does Ex. 18 illustrate the danger of generalizing from too few cases? Explain as fully as you can.

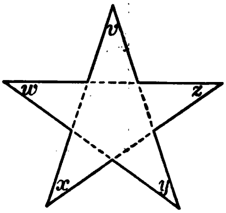


FIG. 166

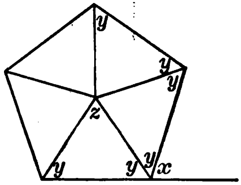


FIG. 167

147. Designs. The triangle, parallelogram, rectangle, square, and many regular polygons are made the basis of a large number of decorative designs (see Fig. 168). The following exercises will bring out some of the most interesting facts connected with designs.

EXERCISES

1. What equiangular (regular) polygons can be put together to make a mosaic pavement, a floor, or a design for linoleum?

2. How many equiangular triangles must be placed together at a common vertex to make a perigon?

3. Show how a tile floor is laid with tiles which have the form of a regular hexagon. How many such tiles must be placed with a common vertex to cover the entire angle about a point?

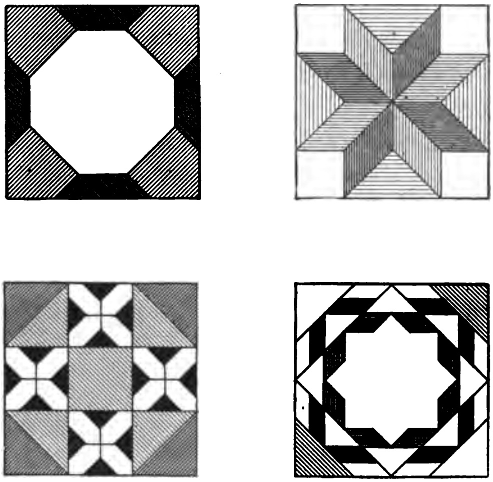


FIG. 168

4. Why does a bee always build the cells in its comb in the form of regular hexagons?

5. Find and bring to class drawings or samples of patterns for tile floors or linoleums which show how regular polygons are used as the basis of the designs.

148. Convex. A polygon each angle of which is less than a straight angle is called a *convex polygon.* Thus, Fig. 169, I, shows a convex polygon. A polygon in which any one of its angles is greater than a straight angle is called a *reëntrant polygon.* Such a polygon is shown in Fig. 169, II and III. We shall be concerned in this course with the convex polygon only.

149. Classification of quadrilaterals. The diagram on page 106 will help the student to understand the names and general appearance of the quadrilaterals which are studied in this chapter. The conditions which determine the type of quadrilateral are given for each figure, and the student should be certain that he understands each case thoroughly.

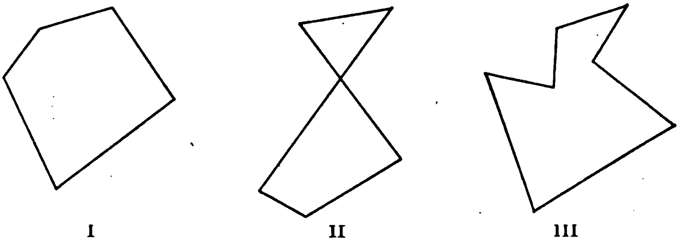


FIG. 169. QUADRILATERALS

150. How the parallelogram is used. Many practical uses are made of parallelograms and parallel lines in designing, surveying, physics, certain types of construction work, and so on. We shall show how the parallelogram is used in some of these fields and shall leave it for the student to seek further applications.

EXERCISES

1. Fig. 170 shows an instrument which can be used for leveling. It consists of a rectangular framework $ABCD$ on which a plumb bob is hung at P, the midpoint of AD. The point M is the midpoint of BC. If when the lower side BC is placed upon the surface to be leveled, the plumb line hangs exactly over M, the surface is level. Explain as fully as you can why this is so, stating the principles involved.

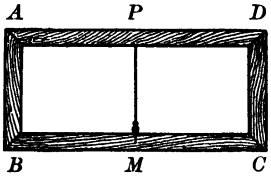


FIG. 170

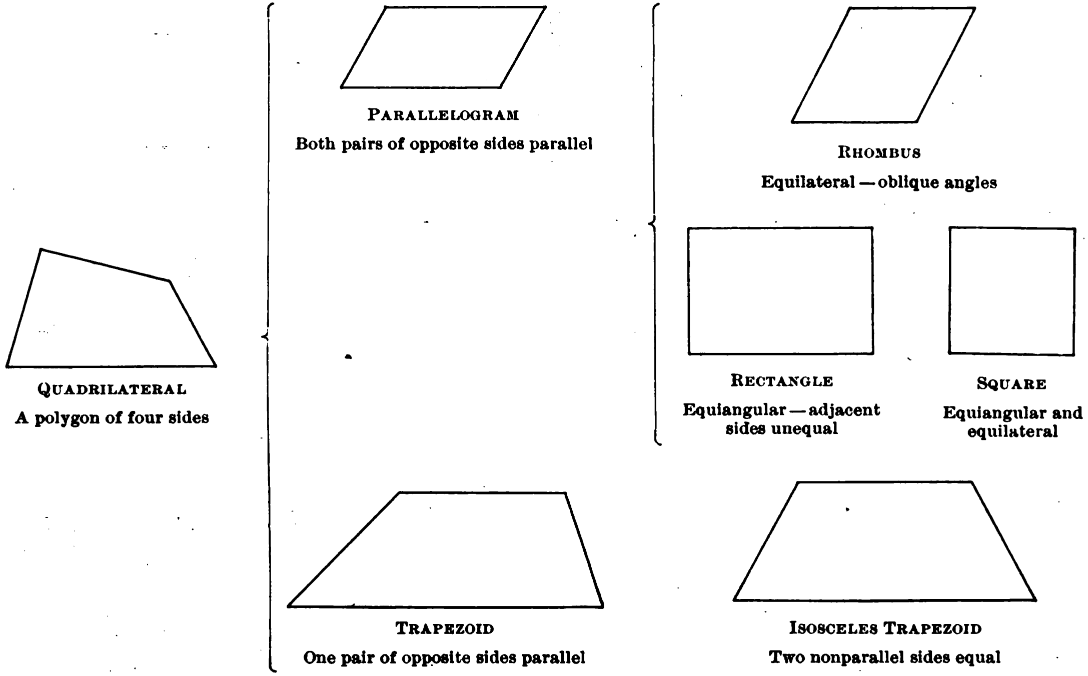

QUADRILATERAL
A polygon of four sides
PARALLELOGRAM
Both pairs of opposite sides parallel
RHOMBUS
Equilateral — oblique angles
RECTANGLE
Equiangular — adjacent sides unequal
SQUARE
Equiangular and equilateral
TRAPEZOID
One pair of opposite sides parallel
ISOSCELES TRAPEZOID
Two nonparallel sides equal

FIG. 171. TYPES OF QUADRILATERALS

2. Fig. 172 shows a device known as the *parallel ruler*, which is used for drawing parallel lines. It is made of two rulers, AD and BC, and two crossbars, AB and DC, hinged as shown. The distances AD and BC are equal, and the same is true of the distances AB and DC. Show that $AD \parallel BC$ for all the various possible positions, and show how the instrument should be used in drawing parallel lines.

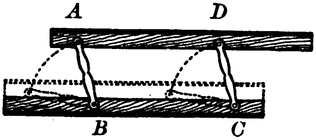


FIG. 172. PARALLEL RULER

3. Fig. 173 shows an instrument called a *gauge*, which is used by carpenters to mark a line parallel to the edge of a board. The gauge has a marking point and a sliding head which is adjusted by a screw as shown. Explain how the instrument may be used.

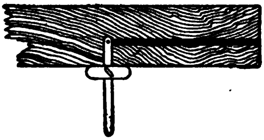


FIG. 173. GAUGE

4. Bring to class descriptive material, designs, or pictures which show other ways in which parallelograms are practically used.

5. How is the parallelogram applied in the type of telephone extensions often found in offices? Bring to class any pictures which illustrate this type of office telephone and explain them.

6. How many different ways can you remember for drawing parallel lines? Describe each briefly.

7. Study Fig. 171 carefully and see how many practical uses you have seen made of the various figures shown. In each case explain why each figure was used.

NOTE. We shall not attempt here to give a larger number of practical uses of parallel lines and parallelograms. However, the student will find that in the subsequent work of this course many other applications are given.

151. Theorem. *A diagonal divides a parallelogram into two congruent triangles.*

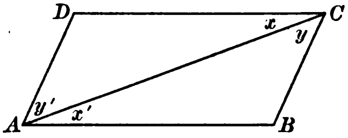


FIG. 174

Given the parallelogram *ABCD*, with the diagonal *AC*.

To prove that $\triangle ABC \cong \triangle ACD$.

Proof

STATEMENTS	REASONS
1. In the $\triangle ABC$ and ACD, $DC \parallel AB$ and $AD \parallel BC$.	1. By definition of a parallelogram.
2. $\angle x = \angle x'$, and $\angle y = \angle y'$.	2. When two parallel lines are cut by a transversal, the alternate-interior angles are equal.
3. $AC \equiv AC$.	3. Obvious.
4. $\therefore \triangle ABC \cong \triangle ACD$.	4. If two angles and the included side of one triangle are equal respectively to two angles and the included side of another triangle, the triangles are congruent.

152. Corollary 1. *The opposite sides of a parallelogram are equal.*

153. Corollary 2. *The opposite angles of a parallelogram are equal* (see also Art. 83, Ex. 6).

154. Corollary 3. *Segments of parallel lines cut off by two other parallel lines are equal.*

155. Corollary 4. *If one of the angles of a parallelogram is a right angle, all the angles are right angles.*

156. Corollary 5. *Parallel lines are everywhere equidistant from each other.*

SUGGESTION. Assume perpendiculars between the parallel lines.

157. Corollary 6. *The sides of a square are equal each to each.*

158. Corollary 7. *The sides of a rhombus are equal each to each.*

159. Theorem. *The diagonals of a parallelogram bisect each other.*

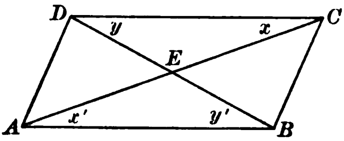


FIG. 175

Given the parallelogram *ABCD*, with diagonals *AC* and *BD*.

To prove that $AE = EC$ *and* $DE = EB$.

Proof

STATEMENTS	REASONS
1. In the $\triangle ABE$ and DEC, $AB = DC$.	1. Opposite sides of a parallelogram are equal.
2. $AB \parallel DC$ and $AD \parallel BC$.	2. By definition of a parallelogram.
3. $\angle x = \angle x'$, and $\angle y = \angle y'$.	3. If two parallel lines are cut by a transversal, the alternate-interior angles are equal.
4. $\therefore \triangle ABE \cong \triangle DEC$.	4. If two angles and the included side of one triangle are equal respectively to two angles and the included side of another triangle, the triangles are congruent.
5. $\therefore AE = EC$, and $DE = EB$.	5. Homologous parts of congruent triangles are equal.

NOTE. The point of intersection of the diagonals of a parallelogram is called the *center* of the parallelogram.

160. *Corollary.* *The diagonals of a square or rhombus bisect each other at right angles.*

EXERCISES

1. The diagonals of a parallelogram divide each other so that the segments of one are $x^2 + 3x$ and $2(6x - 7)$, and the segments of the other are $y^2 + 6y$ and $8(y - 6)$. Find x, y, and the length of each diagonal.

2. In Fig. 176 AB and CD bisect each other at O. Prove that AD is equal and parallel to CB.

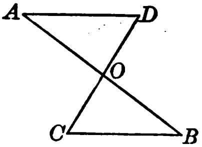


FIG. 176

3. In Fig. 177 $ABCD$ and $EFGH$ are each parallelograms. Prove that the parallelograms are concentric; that is, prove that they have the same center.

SUGGESTION. Draw AC and DB, meeting each other at O. Then draw EO, OG, FO, and OH, and prove that EOG and FOH are straight lines.

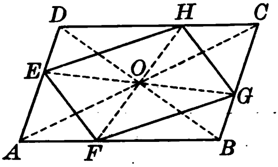


FIG. 177

4. The perpendiculars drawn to the diagonals of a parallelogram from opposite vertices are equal.

5. The bisectors of two adjacent angles of a parallelogram bisect each other at right angles.

161. Summary of properties of a parallelogram. The student will observe that in the theorems on the two preceding pages we have proved the following properties of a parallelogram:

1. The opposite sides are equal.
2. The opposite angles are equal.
3. The consecutive angles are supplementary.
4. The diagonals bisect each other.

EXERCISES

1. If one angle of a parallelogram is 98°, how large is each of the other angles ?

2. If a line through the center of a parallelogram is terminated by two opposite sides, the line is bisected by the center.

3. Two opposite sides of a parallelogram are denoted by x^2+6x and $5(x+4)$. Find the length of each of the given sides.

4. Two opposite angles of a parallelogram are denoted by $2x^2+12$ and $14x+28$. Find x and the size of each angle of the parallelogram.

5. Solve the following quadratic equations by the method of factoring or by the method of completing the square:

(a) $6x^2 = 23x + 4.$ (d) $y^2 + 6y = 1.$

(b) $\dfrac{x}{x-3} - \dfrac{x-1}{x+3} = 2.$ (e) $\dfrac{4y}{y-1} + \dfrac{y-10}{y} = 4.$

(c) $x^2 - 6x = 91.$ (f) $19 - y = 4y^2.$

162. ***Theorem.*** *A quadrilateral is a parallelogram if*

1. *The opposite sides are equal.*
2. *The opposite angles are equal.*
3. *In one pair the sides are both equal and parallel.*
4. *The diagonals bisect each other.*
5. *The consecutive angles are supplementary.*

Proof. The proof is left to the student.

EXERCISES

1. In the five preceding theorems see if you can pick out those which have converses. Give the statements of the converse theorems.

2. The diagonals of a rectangle which is not a square are equal.

3. If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

4. If two adjacent sides of a parallelogram are equal, all the sides are equal.

5. If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is either a rhombus or a square.

6. The diagonals of a rhombus or those of a square bisect the angles through which they pass.

7. State and prove the converse of the corollary in Art. 160.

8. Make a classification of quadrilaterals according as their diagonals

(a) are equal;
(b) are perpendicular to each other;
(c) bisect each other;
(d) bisect the angles through which they pass.

9. If in the parallelogram $ABCD$ a line is drawn through the point of intersection of the diagonals, certain pairs of congruent triangles are formed. Select the congruent pairs and prove their congruence.

10. Prove that the line drawn through the intersection of the diagonals in Ex. 9 bisects the parallelogram $ABCD$.

NOTE. The point of intersection of the diagonals of a parallelogram is the center of gravity.

163. ***Construction problem.*** *To construct a parallelogram when two adjacent sides and the included angle are given.*

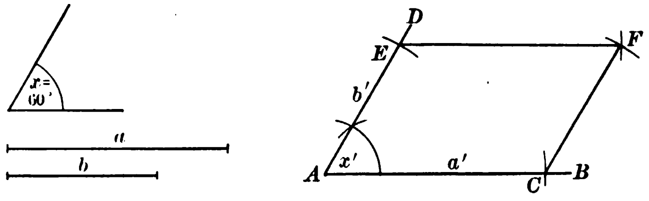


FIG. 178

Given the line segments *a* and *b* and the angle *x*.

Required *to construct a parallelogram with two adjacent sides equal respectively to a and b and the included angle equal to x.*

NOTE. The construction and proof are to be given by the student.

164. Theorem. *Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

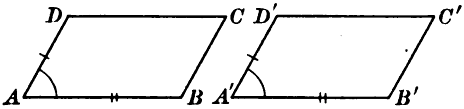


FIG. 179

Given the parallelograms *ABCD* and *A'B'C'D'*, with *AB* equal to *A'B'*, *AD* equal to *A'D'*, and the angle *A* equal to the angle *A'*.

To prove that *parallelogram ABCD is congruent to parallelogram A'B'C'D'.*

Proof. The proof is left to the student.

EXERCISES

1. Two parallelograms are congruent if they have

(a) two sides and the included diagonal of one equal respectively to two sides and the included diagonal of the other;

(b) a side and the diagonal of one equal respectively to a side and the diagonal of the other;

(c) a diagonal and the angle between this diagonal and the adjacent sides of one equal respectively to a diagonal and the angle between this diagonal and the adjacent sides of the other.

2. Given a diagonal of a square, to construct the square.

3. Given the diagonals of a rhombus, to construct the rhombus.

4. Prove that the angles at the ends of either base of an isosceles trapezoid are equal.

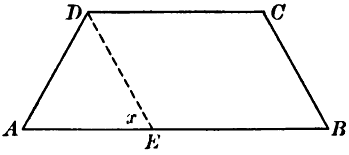


FIG. 180

SUGGESTION. Draw $DE \parallel CB$, as in Fig. 180, and prove $\angle A = \angle B$.

5. If the angles at the ends of either base of a trapezoid are equal, the trapezoid is isosceles.

SUGGESTION. Use Fig. 180.

6. If two parallel lines (Fig. 181) are cut by a transversal, the bisectors of the alternate-interior angles form a rectangle.

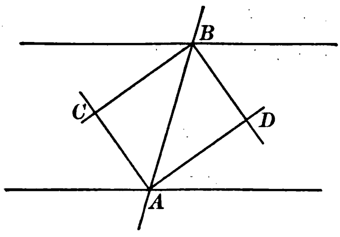


FIG. 181

165. Resultant of two forces acting in a straight line. The *resultant* of two forces is defined as that single force which will produce the same effect upon a body as is produced by the joint action of the two forces. If two spring balances are fastened to a small ring and pulled in the same direction, it will be found that a third spring balance, attached to the same ring and pulled in the opposite direction, will, when equilibrium exists, register exactly the same amount as the sum of the other two. In other words, the resultant of two similarly directed forces applied at the same point is equal to the sum of the two forces. In like manner, the resultant of two oppositely directed forces applied at the same point is equal to the difference between them, and its direction is that of the greater force.

166. Equilibrant. A force like that represented by the third spring balance in Art. 165 is called an *equilibrant.* Thus, *the equilibrant of a force or of forces is that single force which will just prevent the motion which the given force or forces tend to produce.* It is equal and opposite to the resultant.

167. Law of the parallelogram of forces. We find an important application of parallelograms in science. If a body is moved from one place to another in a plane and we wish to describe its displacement, we must tell not only how far the body has been moved but also the direction of the displacement. For example, if we let a point P be displaced from A to B (Fig. 182), we may represent graphically the magnitude and direction of the displacement by the line segment AB, which is called a *directed line segment.* If we let the point P be displaced from A to D, the magnitude and direction of this second displacement is represented by AD. If we let the point P start at A and suffer both of the preceding displacements, either in the above order or in reverse order, P will finally be found at C which is the end of the diagonal of a parallelogram determined by the two directed line segments AB and AD. The diagonal AC, which represents the actual displacement, is the resultant of AB and AD. Thus, the two displacements have the effect of the one displacement represented by the resultant.

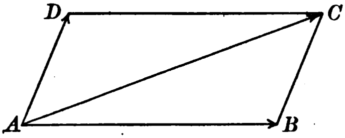


FIG. 182

It is not necessary that the two displacements occur in succession. The displacements often occur simultaneously, as in the case of a man walking across the deck of a moving steamer. The actual displacement of the man's body is the resultant of the displacement due to the motion of the steamer and that due to his walking.

The preceding fact may be stated in the form of a law which in science can be proved by actual experiment.[1] This law, called *the law of the parallelogram of forces* (also called "parallelogram of velocities") is as follows: *If two forces acting at an angle to each other upon a point are represented as to magnitude and direction by two line segments, the resultant is represented as to direction and magnitude by*

---

[1] See Millikan and Gale, Practical Physics, p. 61.

the diagonal of the parallelogram of which the two line segments are adjacent sides. The adjacent sides are called the *components* of the resultant.

168. Vector. Displacements, forces, and velocities are all directed quantities which we can combine by the parallelogram law. Such quantities are called *vectors.* If we choose our units properly, all vectors may be represented by directed line segments, which in turn represent displacements.

We can therefore represent forces by directed line segments, provided the directions of the line segments are the same as the directions of the forces and provided the line segments contain as many length units as there are force units in the magnitudes of the corresponding forces.

EXERCISES

1. A force of 150 lb. and another of 250 lb. are exerted upon an object at an angle of 50° to each other. Represent 10 lb. by a line 1 cm. long, draw the forces to scale, and find the resultant in pounds.

2. A ship is moving east at the rate of 18 mi. an hour, and at the same time the wind is forcing the ship north at the rate of 4 mi. an hour. In what direction and at what rate is the ship traveling?

3. A man starts his motor boat at the rate of 8 mi. an hour directly across a river which is flowing at the rate of 6 mi. per hour. Show that the diagram in Fig. 183 represents the course of the boat.

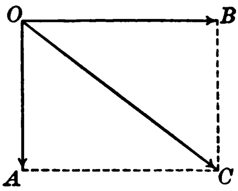


FIG. 183

4. While a steamer sails due north at the rate of 10 mi. per hour, a passenger walks on deck from east to west at the rate of 4 mi. per hour. In what direction and at what speed does the passenger move?

5. A force of 250 lb. is resolved into two components which make angles of 30° and 28° respectively with the direction of the original force. Find the magnitude of each component.

169. Summary of methods of proof. The student will find the following outline of methods of proving certain important relations very useful:

I. *Methods of proving triangles congruent.* We shall first consider the general triangle and then the right triangle.

A. Two triangles are congruent if

1. Two sides and the included angle of one are equal respectively to two sides and the included angle of the other.
2. Two angles and the included side of one are equal respectively to two angles and the included side of the other.
3. Three sides of one are equal respectively to three sides of the other.
4. Two angles and any side of one are equal respectively to two angles and any side of the other.

B. Two right triangles are congruent if

1. The hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.
2. The hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.
3. A side and any acute angle of one are equal respectively to a side and any acute angle of the other.

II. *Methods of proving angles equal.* Two angles are equal if they

1. Are homologous angles of congruent triangles.
2. Are supplements or complements of the same angles or equal angles.
3. Are right angles or vertical angles.
4. Are opposite the equal sides of an isosceles triangle.
5. Are opposite the equal sides of an isosceles trapezoid.
6. Are alternate-interior angles or corresponding angles made by a transversal cutting two parallel lines.
7. Have their sides respectively parallel or perpendicular.
8. Are opposite angles of a parallelogram.

III. *Methods of proving line segments equal.* Two line segments are equal if they are

1. Homologous sides of congruent triangles.
2. Opposite sides of a parallelogram.
3. Opposite equal angles in a triangle.
4. Sides of a square or of a rhombus.

IV. *Methods of proving lines parallel.* Two lines are parallel if

1. The alternate-interior angles made by a transversal are equal.
2. The corresponding angles made by a transversal are equal.
3. The interior angles on the same side of the transversal are supplementary.

4. They are parallel to, or perpendicular to, the same line.
5. They are opposite sides of a parallelogram.
6. They are bases of a trapezoid.

170. Prismatic surface. If in a given polygon $ABCDE$ (Fig. 184) we let a straight line AR, not in the plane of the polygon, move so as always to be parallel to its original position AR and so as always to touch the polygon, the line AR is said to generate a *prismatic surface*. Such a surface is shown in Fig. 185.

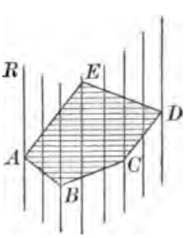


FIG. 184

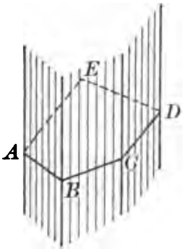


FIG. 185

171. Prismatic space. The space inclosed by a prismatic surface is called *prismatic space.*

172. Prism. If a given polygon $ABCDE$ (Fig. 186) moves to the position $A'B'C'D'E'$ and in so doing remains parallel to its original position, the points A, B, C, D, and E moving along the lines AA', BB', CC', DD', and EE', that portion of the prismatic space inclosed by the prismatic surface and the planes of the two polygons is called a *prism.*

173. Bases. The surfaces inclosed by the boundaries of the polygons $ABCDE$ and $A'B'C'D'E'$ (Fig. 186) are called the *bases* of the prism.

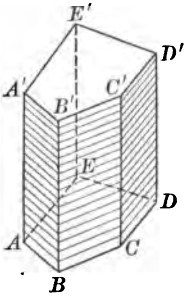


FIG. 186

174. Lateral surface; lateral faces. That portion of a prismatic surface cut off by the bases of a prism is called the *lateral surface* of the prism. The quadrilaterals which make up the lateral surface are called the *lateral faces* of the prism.

175. Total surface. The *total surface* of a prism is made up of the lateral surface and the surfaces of the two parallel bases of the prism.

EXERCISES

1. Prove that the lateral faces of a prism are parallelograms.

2. How can a triangular prism be formed? a quadrangular prism?

3. What relation must the generating line AR (the generatrix) bear to the plane of the given polygon (Fig. 184) if the lateral faces of the prism are to be rectangles?

4. What special name do we give to a prism the lateral faces and bases of which are rectangles?

176. *Theorem.* *If a line is drawn from the midpoint of one side of a triangle to the midpoint of another side, the line is parallel to the third side and equal to one half of it.*

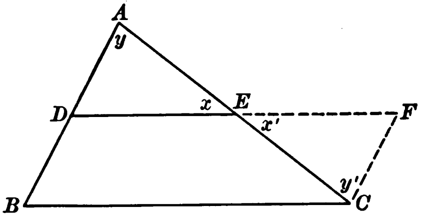


FIG. 187

Given the triangle *ABC*, with *D* the midpoint of *AB*, and *E* the midpoint of *AC*.

To prove that $DE \parallel BC$ *and* $DE = \frac{1}{2}BC$.

Proof

STATEMENTS	REASONS
1. Extend DE to F, making $EF = DE$.	1. A line can be drawn equal to another line.
2. Draw CF.	2. One and only one straight line can be drawn between two points.
3. If we can show that $BCFD$ is a parallelogram, then the theorem is proved.	3. Opposite sides of a parallelogram are equal, and since $DE = \frac{1}{2} DF$, if $DF = BC$, then $DE = \frac{1}{2} BC$.
4. In the $\triangle$s ADE and ECF, $DE = EF$.	4. By construction.
5. $\angle x = \angle x'$.	5. Vertical angles are equal.
6. $AE = EC$.	6. Given.
7. $\therefore \triangle ADE \cong \triangle ECF$.	7. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.
8. $\therefore FC = AD$ and $\angle y = \angle y'$.	8. Homologous parts of congruent triangles are equal.
9. $\therefore FC \parallel AB$.	9. If the alternate-interior angles formed by two lines and a transversal are equal, the lines are parallel.
10. $AD = BD$.	10. Given.
11. $\therefore FC = BD$.	11. Things equal to the same thing are equal to each other.
12. $\therefore BCFD$ is a parallelogram.	12. If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.
13. $\therefore DE \parallel BC$.	13. By definition of a parallelogram.
14. Also $DF = BC$.	14. Opposite sides of a parallelogram are equal.
15. $\therefore DE = \frac{1}{2} DF = \frac{1}{2} BC$.	15. $DE = \frac{1}{2} DF$ by construction, and $DF = BC$ by 14.

177. *Theorem.* *A line that bisects one side of a triangle and is parallel to a second side bisects the third side and is equal to one half the second side.*

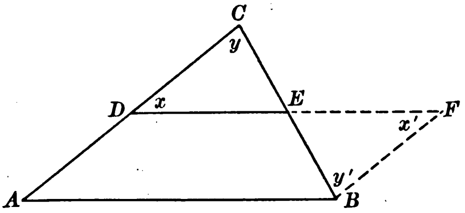


FIG. 188

Given the triangle *ABC*, with *AD* equal to *DC*, and *DE* parallel to *AB*.

To prove that $CE = EB$ *and* $DE = \frac{1}{2}AB$.

Proof

STATEMENTS	REASONS
1. Draw $BF \parallel AC$, and produce DE to F.	1. Through a point not on a given line one and only one line can be drawn parallel to the given line.
2. $DF \parallel AB$.	2. Given.
3. Also, $BF \parallel AD$.	3. By construction.
4. $\therefore ABFD$ is a parallelogram.	4. By definition.
5. Then $BF = AD$.	5. Opposite sides of a parallelogram are equal.
6. But $AD = DC$.	6. Given.
7. $\therefore BF = DC$.	7. Things equal to the same thing are equal to each other.
8. $\angle x = \angle x'$ and $\angle y = \angle y'$.	8. If two parallel lines are cut by a transversal, the corresponding angles are equal.
9. $\therefore \triangle DCE \cong \triangle BFE$.	9. Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.

Proof (*Continued*)

STATEMENTS	REASONS
10. $\therefore CE = EB$.	10. Homologous parts of congruent triangles are equal.
11. Also, $DF = AB$.	11. Opposite sides of a parallelogram are equal.
12. And $DE = EF$.	12. Homologous parts of congruent triangles are equal.
13. $\therefore DE = \frac{1}{2} DF = \frac{1}{2} AB$.	13. Obvious.

EXERCISES

1. In Fig. 188 show that $\frac{CD}{DA} = \frac{CE}{EB}$.

2. In Ex. 1, if $DA = 5$, $CD = 5$, $CE = 2x + 4$, and $EB = 3x - 2$, find the lengths of CE and EB.

3. The line segments which connect the midpoints of adjacent sides of any quadrilateral form a parallelogram (use Fig. 189).

4. The line segments which connect the midpoints of opposite sides of any quadrilateral bisect each other.

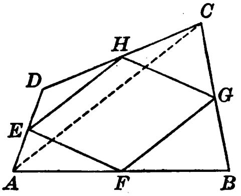


FIG. 189

5. The lines joining the midpoints of the sides of a triangle divide it into four congruent triangles (use Fig. 190).

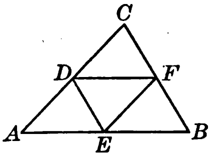


FIG. 190

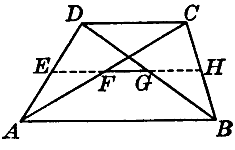


FIG. 191

6. In Fig. 191 $ABCD$ is a trapezoid, and F and G are the midpoints of AC and DB respectively. Prove that $FG = \frac{AB - DC}{2}$.

SUGGESTION. Apply the theorems of Arts. 176 and 177.

178. Theorem. *If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on any other transversal.*

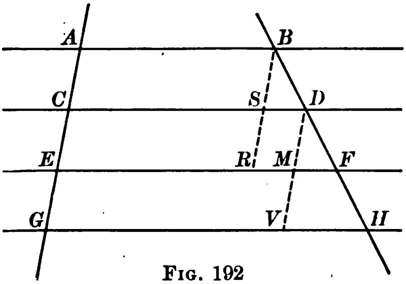


Fig. 192

Given the parallel lines AB, CD, EF, and GH cutting off the equal segments AC, CE, and EG on the transversal AG.

To prove that *on the transversal BH, $BD = DF = FH$.*

Proof

Statements	Reasons
1. Draw $BR \parallel AE$.	1. Through a point not on a given line one and only one line can be drawn parallel to the given line.
2. Then $ACSB$ and $CERS$ are parallelograms.	2. By definition.
3. Also, $AC = BS$, and $CE = SR$.	3. Opposite sides of a parallelogram are equal.
4. But $AC = CE$.	4. Given.
5. $\therefore BS = SR$.	5. Things equal to equal things are equal to each other.
6. Then, in the $\triangle BRF$, $BD = DF$.	6. If a line is drawn from the midpoint of one side of a triangle parallel to a second side, it bisects the third side.
7. In like manner, by drawing $DV \parallel CG$ we have $DF = FH$.	7. Obvious from what is proved above.

EXERCISES

1. Divide a given line segment into four equal parts by using the theorem above. Prove that your construction is correct.

2. There is an easier and a shorter way than that used in Ex. 1 to divide a line segment into any required number of equal parts. To divide the line segment *AB* in Fig. 193, draw *AC* ∥ *BD*, and lay off the same number of equal segments on *AC* and *BD*. Join the corresponding points, and the line *AB* is divided as required. Prove that the construction is correct.

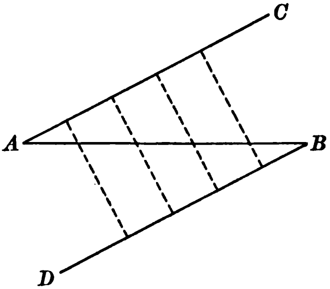


Fig. 193

3. How can a sheet of ruled paper be used to divide a given line segment into equal parts (see Fig. 194)?

4. To construct an equilateral triangle when the perimeter is given.

5. To construct a square when the perimeter is given.

6. Is there any method of dividing a line segment into an even number of equal parts that will not work for an odd number of equal parts? Explain your answer.

7. How may a carpenter lay off the lines so that he may saw an 11-inch board into four strips of equal width? Prove your answer.

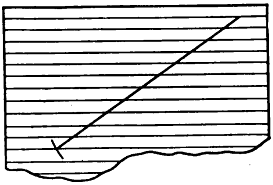


Fig. 194

8. The line segment which connects the midpoints of adjacent sides of a rhombus forms a triangle with the sides.

9. The line segments which connect the midpoints of adjacent sides of a square form a square.

10. The line segments which connect the midpoints of adjacent sides of a rectangle that is not a square form a rhombus.

11. In the parallelogram $ABCD$ in Fig. 195 M and N are the midpoints of DC and AB respectively. Prove that DN and BM trisect the diagonal AC.

12. Construct an equilateral triangle of perimeter 8 in.

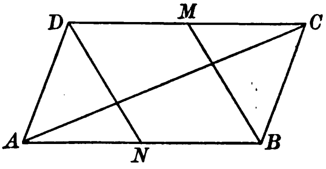


FIG. 195

179. ***Theorem.*** *The line segment which connects the midpoints of the nonparallel sides of a trapezoid is parallel to the bases and equal to one half the sum of the bases.*

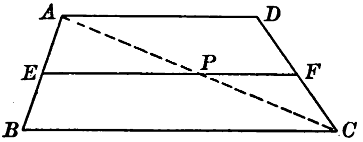


FIG. 196

Given the trapezoid *ABCD*, with *AE* equal to *EB*, and *DF* equal to *FC*.

To prove that $EF \parallel AD$, $EF \parallel BC$, *and* $EF = \frac{1}{2}(AD + BC)$.

Proof

STATEMENTS	REASONS
1. Draw AC. Then take P as the midpoint of AC and draw PE and PF.	1. One and only one straight line can be drawn between two points.
2. $EP \parallel BC$, and $EP = \frac{1}{2} BC$.	2. If a line is drawn from the midpoint of one side of a triangle to the midpoint of a second side, the line is parallel to the third side and equal to one half of it.
3. In like manner, $PF \parallel AD$, and $PF = \frac{1}{2} AD$.	3. For the same reason.

Proof (*Continued*)

STATEMENTS	REASONS
4. $\therefore EP+PF=EF$ $=\frac{1}{2}(AD+BC)$.	4. If equals are added to equals, the sums are equal.
5. Since EP and PF both pass through the point P and are $\parallel$ BC and AD respectively, EPF must be a straight line $\parallel$ AD and BC.	5. Through a point outside a given line one and only one straight line can be drawn parallel to the given line.

180. *Corollary.* *A line which bisects one of the nonparallel sides of a trapezoid and is parallel to the bases bisects the other side.*

EXERCISES

1. Construct a trapezoid given the following: lower base, 8 in.; a nonparallel side, 5 in.; the angle between these sides, 50°.

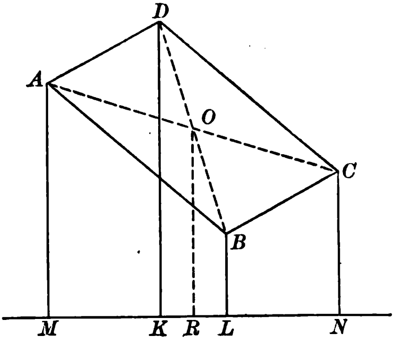


FIG. 197

2. If the angles at the ends of one base of a trapezoid are equal, the angles at the ends of the other base are equal.

3. The opposite angles of a trapezoid are supplementary.

4. The diagonals of an isosceles trapezoid are equal.

5. If a trapezoid has equal diagonals the trapezoid is isosceles.

SUGGESTION. Draw perpendiculars from the ends of the upper base upon the lower base.

6. To construct a trapezoid when the four sides are given.

7. To construct a trapezoid when the two bases and the two base angles are given.

8. If a parallelogram $ABCD$ is drawn above a given line MN, as shown in Fig. 197, and if the perpendiculars AM, DK, BL, and CN are drawn to MN, then the sum of the perpendiculars AM and CN equals the sum of the perpendiculars DK and BL.

Suggestion. Draw the diagonals AC and BD and draw $OR \perp MN$.

9. The sum of the distances of the three vertices of a triangle from any straight line outside the triangle is equal to the sum of the distances of the midpoints of the sides from that line (see Fig. 198).

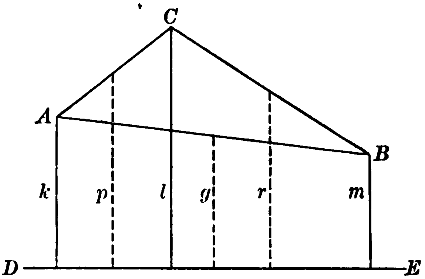


Fig. 198

10. Using Fig. 199, show that

(a) A trapezoid may be divided into two triangles (Fig. 199, I).

(b) A trapezoid may be divided into two right triangles and a rectangle (Fig. 199, II).

(c) If a trapezoid is isosceles, it may be divided into two congruent right triangles and a rectangle (Fig. 199, III).

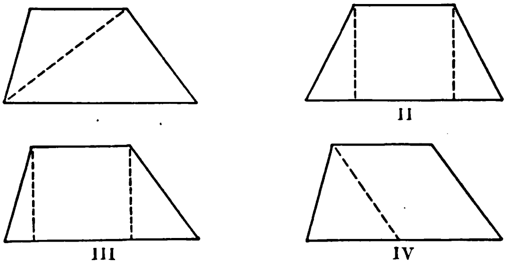


Fig. 199

(d) A trapezoid may be divided into a parallelogram and a triangle (Fig. 199, IV).

181. Inequalities. As you have already learned (Art. 21), an inequality is a statement that two numbers or magnitudes are unequal. Thus, $a > b$ (read "a is greater than b")

and $a < b$ (read "a is less than b") are inequalities. If we are not particularly concerned as to which is the greater, we write $a \neq b$ (read "a is not equal to b").

We say that $a > b$ and $x > y$ are *inequalities of the same order*, and that $a > b$ and $x < y$ are *inequalities of the reverse order*.

182. Axioms used in combining inequalities. The following axioms are to be used in combining inequalities:

1. *If unequals are added to equals, the sums are unequals of the same order as the unequal addends.*

Thus, if $c = c$ and $a < b$, then $a + c < b + c$.

2. *If equals are subtracted from unequals, the differences are unequals of the same order as the unequal minuends.*

Thus, if $c = c$ and $a < b$, then $a - c < b - c$.

3. *If unequals are added to unequals of the same order, the sums are unequals of the same order.*

Thus, if $a < b$ and $c < d$, then $a + c < b + d$.

4. *If unequals are subtracted from equals or from unequals of the opposite order, the differences are unequals of the order opposite to that of the subtrahend.*

Thus, if $a > b$ and $c < d$, then $a - c > b - d$.

5. *If unequals are multiplied by positive equals, the products are unequals of the same order as the multiplicands.*

Thus, if $a < b$ and $c = c$, then $ac < bc$.

6. *If unequals are multiplied by negative equals, the products are unequals of the order opposite to that of the multiplicands.*

Thus, if $a < b$ and $-c = -c$, then $-ac > -bc$.

7. *If unequals are divided by positive equals, the quotients are unequals of the same order as the dividends.*

Thus, if $a < b$ and $c = c$, then $\frac{a}{c} < \frac{b}{c}$.

8. *If unequals are divided by negative equals, the quotients are unequals of the order opposite to that of the dividends.*

Thus, if $a < b$ and $-c = -c$, then $\frac{-a}{c} > \frac{-b}{c}$.

9. *If* $a > b$ *and* $b \geqq c$, *then* $a > c$; *or if* $a \geqq b$ *and* $b > c$, *then* $a > c$.

NOTE. The symbol $\geqq$ is read "equal to or greater than."

EXERCISE

Show the truth of each of the preceding axioms by giving an arithmetical example.

183. Fundamental inequality relations. The following fundamental relations of inequality are important, and the student should keep them in mind:

1. *The sum of two sides of a triangle is greater than the third side.*

This follows at once from postulate 4 in Art. 7; that is, in Fig. 200,

$$AC + BC > AB.$$

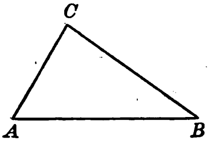


FIG. 200

2. *Any side of a triangle is greater than the difference between the other two sides.*

From 1 we have the inequality

$$AC + BC > AB.$$

Subtracting BC from both members, we have

$$AC > AB - BC.$$

3. *A straight line is the shortest distance between two points.*

4. *An exterior angle of a triangle is greater than either nonadjacent interior angle.*

NOTE. The student should show why Nos. 3 and 4 are true.

184. Theorem. *If two angles of a triangle are unequal, the sides opposite them are unequal, the greater side lying opposite the greater angle.*

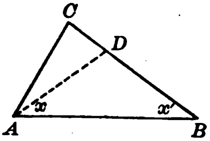


FIG. 201

Given the triangle *ABC*, with angle *A* greater than angle *B*.

To prove that $BC > AC$.

Proof

STATEMENTS	REASONS
1. Since $\angle A > \angle B$ we can lay off $\angle B$ on $\angle A$; that is, we can make $\angle X = \angle B$.	1. Obvious.
2. Then $AD = DB$.	2. If two angles of a triangle are equal, the sides opposite these angles are equal.
3. Also $AD + DC > AC$.	3. The sum of two sides of a triangle is greater than the third side.
4. $\therefore BD + DC > AC$.	4. By substituting BD for AD.
5. $\therefore BC > AC$.	5. Since BC obviously equals $BD + DC$.

185. Corollary. *The hypotenuse of a right triangle is greater than either leg of the triangle.*

186. Theorem. *If two sides of a triangle are unequal, the angles opposite them are unequal, the greater angle lying opposite the greater side.*

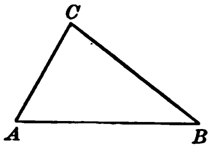


Fig. 202

Given the triangle *ABC*, with *BC* greater than *AC*.

To prove that $\angle A > \angle B$.

Proof

Statements	Reasons
1. There are three possibilities: $\angle A = \angle B$, $\angle A < \angle B$, or $\angle A > \angle B$.	1. Obvious.
2. $\angle A \neq \angle B$.	2. Because, if $\angle A = \angle B$, then the $\triangle ABC$ is isosceles, and this is contrary to the hypothesis.
3. $\angle A \nless \angle B$.	3. Because, if $\angle A < \angle B$, then, by the theorem of Art. 184, $BC < AC$, and this is contrary to the hypothesis.
4. $\therefore \angle A > \angle B$.	4. This is the only remaining possibility.

EXERCISES

1. The shortest distance from a point to a line is the perpendicular from the point to the line.

Suggestion. Let AB (Fig. 203) be the given perpendicular. Then show that $AB < AC$, any other line.

2. Oblique lines from a point on a perpendicular to a line which cut off equal distances from the foot of the perpendicular are equal.

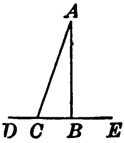


Fig. 203

3. The distance between the centers of two intersecting circles is less than the sum of the radii but greater than the difference.

4. Of two oblique lines drawn to a line from a point on a perpendicular to the line and cutting off on the line unequal segments from the foot of the perpendicular, the more remote is the greater, and conversely.

5. Equal oblique line segments drawn to a line from a point on a perpendicular to the line

(a) Cut off equal distances from the foot of the perpendicular.

(b) Make equal angles with the perpendicular.

(c) Make equal angles with the line.

6. If P is any point in the base AB of an isosceles $\triangle ABC$, then $CP < CB$.

7. Either leg of an isosceles triangle is greater than one half the base.

8. The median to any side of a triangle is less than one half the sum of the other two sides.

SUGGESTION. Extend the median to twice its length, as in Fig. 204.

9. If two sides of a triangle are 15 and 10, between what limiting values does the third side lie?

10. If from a point P within the $\triangle ABC$ lines are drawn from P to the ends of AB, then $AC + CB > AP + PB$.

SUGGESTION. Extend AP until it cuts CB.

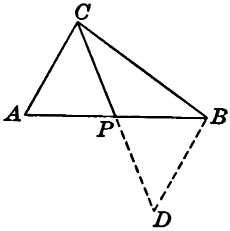


FIG. 204

11. The sum of the three line segments from a point within a triangle to the three vertices is less than the perimeter but greater than the semiperimeter of the triangle.

12. The sum of the diagonals of a quadrilateral is less than the perimeter but greater than the semiperimeter of the quadrilateral.

13. In Fig. 205 C and D represent two towns situated near a river AB. Locate a point P on the bank of the river such that in laying a pipe line from the river to each of the towns the total amount of pipe required may be a minimum.

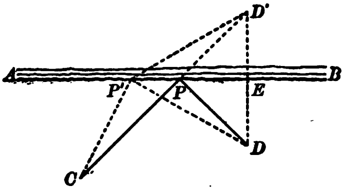


Fig. 205

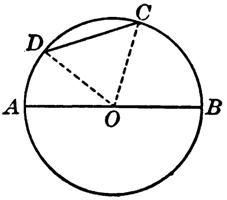


Fig. 206

Suggestion. Draw $DD' \perp AB$, making $DE = ED'$, and draw CD' cutting AB at P. Then show that P is the proper point by taking P' as any other possible point.

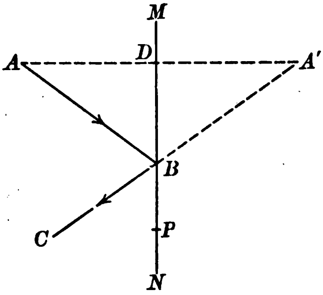


Fig. 207

14. Show that the diameter of a circle (Fig. 206) is greater than any other chord of the circle.

Note. A *chord* of a circle is a line segment which joins two points of the circle.

15. A ray of light starting at A (Fig. 207) is reflected by a plane mirror MN to the point C (see Ex. 2, Art. 142). Locate the point where the ray strikes the mirror.

Suggestion. Draw $AD \perp MN$ and prolong AD to A', making $DA' = AD$. Then connect A' and C.

16. Prove that the path of the ray in Ex. 15, that is, $AB + BC$, is shorter than any other path, say $AP + PC$, P being any other point than B on MN.

17. Fig. 208 shows the top of a billiard table. Show how we may determine a point P such that a ball coming from B_1 and striking the cushion at P will rebound toward another ball at B_3.

SUGGESTION. Use the method of Ex. 15, above.

18. Is the distance $B_1P + PB_3$ (Fig. 208) less than the distance $B_1x + xB_3$, where x is any other point on the cushion than P? Explain your answer.

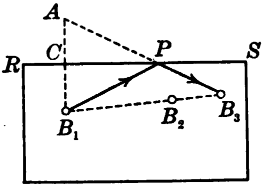


FIG. 208

19. If from any point P in the base of an isosceles $\triangle ABC$ (Fig. 209) perpendiculars are drawn to the equal sides, the sum of their lengths is constant for each position of P and is equal to the perpendicular from either extremity of the base to the opposite side.

SUGGESTION. Draw $PG \parallel AC$ and then prove that $PF + PE = BD$ by showing that $PF = GD$ and that $PE = BG$.

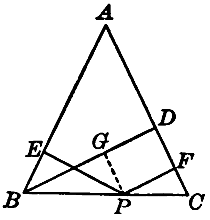


FIG. 209

20. The sum of the three perpendiculars drawn from any point inside an equilateral triangle to the three sides is equal to an altitude.

SUGGESTION. Draw $HI \parallel BC$ (Fig. 210) through P, draw $HJ \perp AC$, and apply Ex. 19.

21. If upon the three sides of any triangle, equilateral triangles are constructed outward and a line is drawn from each vertex of the triangle to the farthest vertex of the equilateral triangle on the opposite side, these three lines are equal.

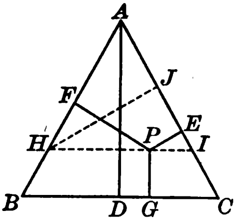


FIG. 210

22. If in Fig. 209 perpendiculars are drawn from any point in the base BC produced upon AB and AC, their difference equals the perpendicular from either end of the base upon the opposite side.

187. Method of proof by elimination. The method of proof used in Art. 186 is called the *method of proof by elimination.* It consists in taking all the possibilities in the situation and showing that all but one of them contradict some known theorem, fact, or assumption. It is usually best to take as the last possibility the thing which you are trying to prove. In the theorem referred to above, for example, we said that $\angle A = \angle B$, $\angle A < \angle B$, or $\angle A > \angle B$, and then proceeded to show that both of the first two possibilities were contrary to fact, leaving the last possibility as the only one.

EXERCISES

1. Show that the theorem of Art. 186 is the converse of the theorem of Art. 184.

2. From a point outside a given line only one perpendicular can be drawn to the line.

3. In bridge construction engineers often use an inverted king-post truss similar to the arrangement shown in Fig. 211. AC and CB are steel bars which are hinged together at C and rest on a metal plate beneath. The plate is held in place by rods. Show why this arrangement will safely support a heavy weight.

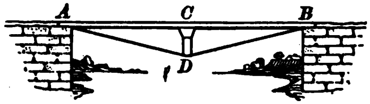


FIG. 211

188. Opposite of a proposition. In the theorem of Art. 186 the hypothesis and the first part of the conclusion are the opposites, or contradictions, of the hypothesis and the conclusion of the theorem of Art. 116. Whenever two theorems are so related they are called *opposites.* The opposite of a truth is not always true; for example, the statement "If a man lives in the city of Minneapolis, he lives in Minnesota" is true, but the opposite statement, "If a man does not live in the city of Minneapolis, he does not live in Minnesota," is not necessarily true.

189. Relation of "converse" and "opposite." If the converse of a proposition is true, its opposite is also true; and if the opposite of a theorem is true, the converse also is true. The truth of the preceding statement may be expressed symbolically as follows:

1. DIRECT STATEMENT. If A is B, then C is D.
2. CONVERSE. If C is D, then A is B.
3. OPPOSITE. If A is not B, then C is not D.

If 2 is true, then 3 is true. In like manner, if 3 is true, then 2 is true.

CHAPTER IV

LOCI OF POINTS

190. Locus. If a point moves according to certain given geometric conditions, it traces out a path which is called the *locus* of the point which satisfies the given conditions. "Locus" is a Latin word meaning *place*; the plural of "locus" is "loci."

If a point moves in a plane so as always to be at a distance of 1 in. from a given point O of the plane, its locus is a circle whose radius is 1 in. and whose center is O. All points in the plane which are 1 in. from O lie on this circle, and, conversely, every point on this circle is 1 in. from O. We may also state this fact as follows: Every point on the circle is 1 in. from O and every point not on the circle is not 1 in. from O. Thus, a circle is the locus (place) of all points in a plane equidistant from a given point of the plane.

From what has been said about symmetry it can be seen that a circle is symmetric about its center and about any one of its diameters.

191. Locus in space. If a point is free to move in space and not simply in a plane, the locus of the point is the surface of a sphere whose center is O.

In order to discover what the locus of all points having a given property is, the student should locate freehand, or by careful construction, a number of points, usually three or more, which possess the given property, and by studying the situation discover what the proper locus is.

For example, if the student wishes to locate a certain point in the classroom which the teacher has in mind, the student must obtain three distinct items of information concerning the point. If the teacher says, "The point which I have in mind is 3 ft. from the east wall of the room," it is evident that there is a plane of points, any one of which is 3 ft. from the east wall. If the teacher adds, "The point is also 8 ft. from the ceiling," then there is still a line of points which satisfy the given conditions. But if the teacher says, "The point is as far from the north wall as it is from the south wall," then and only then can the student tell the exact location of the point which the teacher has in mind.

EXERCISES

1. Make up examples similar to the one above and let your classmates locate the point which you have in mind.

2. Where are all the points in a plane which are 2 in. from a given point of the plane? Where are all these points in space?

3. What is the locus in a plane of all points which are 2 in. from a given line? What is the locus in space?

4. What is the locus in a plane of all points which are equidistant from the ends of a line segment 2 in. long? What is the locus in space?

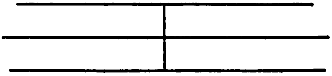


FIG. 212

5. What is the locus in a plane of all points equidistant from two given parallel lines (see Fig. 212)? What is the locus in space?

6. What is the locus in a plane of all points which are equidistant from two given intersecting lines? What is the locus in space?

7. What is the locus of the center of the wheel of a wagon moving along a straight, level road?

8. What is the locus of the center of a silver quarter which rolls around the edge of a silver dollar and always just touches it?

9. What is the locus of all points common to two intersecting planes?

10. Two rulers *AB* and *BC* are placed perpendicular to each other, as shown in Fig. 213. Another ruler *AC* is free to move under the condition that its ends always touch *AB* and *BC*. Find the locus of all possible positions of *P*, the midpoint of *AC*.

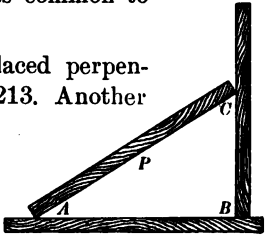


FIG. 213

11. If a circle *O* (Fig. 214) is rolled without sliding along the edge of a ruler *AB*, any point *P* on the circle will trace out a smooth curve called a *cycloid*; that is, the locus of *P* is a cycloid. Cut a circle from stiff cardboard, roll it carefully along a ruler, mark several positions of a chosen point *P*, and draw a smooth curve through the points so marked.

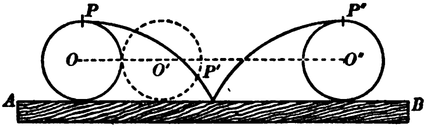


FIG. 214

192. Use of loci. We often find the notion of a locus very useful in locating points which satisfy two or more conditions at the same time.

For example, let us try to find the locus in a plane of a point at a given distance *r* from a fixed point *O* and at a given distance *d* from a given straight line *AB*.

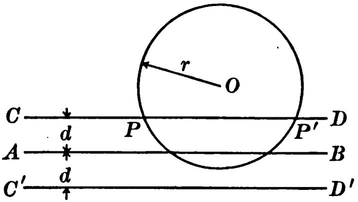


FIG. 215

Solution. Since the required point must be located at a distance *r* from the given point *O*, then it must lie on a circle of radius *r* and whose center is *O* (Fig. 215). And since it must also be at a given distance d from the line AB, the required point must lie on one of the lines CD or $C'D'$, which are on opposite sides of AB, parallel to AB at a distance d from it. If a point such that both of these conditions are satisfied exists, it must lie on the intersection of the two loci; that is, at P and P'.

It is evident that there is no solution of the problem if the loci do not intersect.

193. Method of proving a locus. The following method of attack may be helpful to the student in solving a locus problem:

1. Locate three or more points which satisfy the given conditions, thus suggesting the proper locus.
2. Draw the locus suggested by the points.
3. Prove that the locus you have chosen is correct by proving either (a) and (b) below, or (a) and (c) below.

(a) Every point on the locus satisfies the given conditions.

(b) Every point which satisfies the given conditions lies on the locus.

(c) Every point not on the locus does not satisfy the given conditions.

The student should notice that (b) in the outline above is the converse of (a) and that (c) is the opposite of (a). The opposite of (b), let us call it (d), may be stated as follows: Every point which does not satisfy the given conditions does not lie on the locus. When (a) and (b) are known, then (c) and (d) can be proved by the indirect method; and, in like manner, when (a) and (c) are known, then (b) and (d) can be proved by the indirect method.

The preceding discussion shows that the proof of a locus problem must be based upon a direct proposition and its opposite, or upon a direct proposition and its converse.

194. Theorem. *The locus in a plane of a point equidistant from two given points is the perpendicular bisector of the line segment joining the points.*

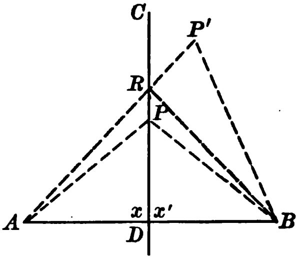


FIG. 216

Given the perpendicular bisector *CD* of the line *AB*.

To prove that *CD is the locus of a point equidistant from A and B.*

Proof

STATEMENTS	REASONS
1. Let P be any point on CD, and draw PA and PB.	1. One and only one straight line can be drawn between two points.
2. $AD = DB$, and $\angle x = \angle x'$.	2. Given.
3. $PD \equiv PD$.	3. Obvious.
4. $\therefore \triangle ADP \cong \triangle DBP$.	4. Two triangles are congruent if two sides and the included angle of one triangle are equal respectively to two sides and the included angle of the other.
5. $\therefore PA = PB$.	5. Homologous parts of congruent triangles are equal.
6. The first test of a locus is satisfied.	6. We have shown that any point P on the perpendicular bisector is equidistant from A and B.

Proof (*Continued*)

STATEMENTS	REASONS
7. Let P' be any point not on CD, and draw $P'A$ and $P'B$.	7. One and only one straight line can be drawn between two points.
8. $P'R + RB > P'B$.	8. The sum of two sides of a triangle is greater than the third side.
9. But $AR = RB$.	9. Shown in 6 above.
10. $\therefore P'R + RA > P'B$.	10. By substitution.
11. $\therefore P'A > P'B$.	11. By substitution.
12. The second test of a locus is satisfied.	12. We have shown that any point P' not on the perpendicular bisector is not equidistant from A and B.
13. $\therefore$ the theorem is true.	

EXERCISES

1. The perpendicular bisector of the base of an isosceles triangle passes through the vertex.

2. What is the locus of the vertices of all isosceles triangles constructed on a given base? Explain your answer.

3. If two points of one line are equidistant from two points of another line, the lines are perpendicular.

4. If a straight line of a constant length moves, with one end constantly on a given line, so as always to make the same angle with the given line and on the same side of it, what is the locus of the other end?

5. If a straight line of constant length moves around a square with one end constantly on the square, and in the same plane with it, so that it is always parallel to its original position, what is the locus of its other end?

6. Answer the question in Ex. 5 if the line of constant length moves around any triangle, quadrilateral, or parallelogram.

195. *Theorem.* *The locus of a point equidistant from the sides of an angle is the bisector of the angle.*

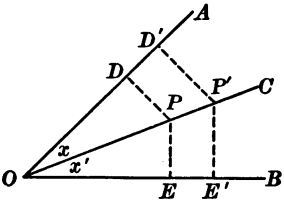


FIG. 217

Given the angle *AOB* and its bisector *OC*.

To prove that *OC is the locus of all points equidistant from OA and OB.*

Proof

STATEMENTS	REASONS
1. From any point P on OC draw PD and PE, each $\perp$ OA and OB respectively.	1. From a point outside a given line one and only one perpendicular can be drawn to the line.
2. In the $\triangle$s OPE and OPD, $OP \equiv OP$, and $\angle x = \angle x'$.	2. Given.
3. Then $\triangle OPE \cong \triangle OPD$.	3. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.
4. $\therefore PD = PE$.	4. Homologous parts of congruent triangles are equal.
5. $\therefore$ any point on the bisector is equidistant from the sides OA and OB.	5. We took P as any point.
6. From any point P' equidistant from OA and OB draw $P'D'$ and $P'E'$ $\perp$ OA and OB respectively.	6. From a point outside a given line one and only one perpendicular can be drawn to the line.

Proof (*Continued*)

STATEMENTS	REASONS
7. In the $\triangle$s $OP'D'$ and $OE'P'$, $OP' \equiv OP'$, and $P'D' = P'E'$.	7. Given.
8. $\therefore \triangle OP'D' \cong \triangle OE'P'$.	8. Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.
9. $\therefore \angle x = \angle x'$.	9. Homologous parts of congruent triangles are equal.
10. $\therefore$ P' lies on the bisector of $\angle AOB$, and the second test of a locus is satisfied.	10. We have shown that any point equidistant from the sides of an angle lies on the bisector.
11. $\therefore$ the theorem is true.	

EXERCISES

1. If two straight lines intersect, the locus of all points equidistant from the two lines is a pair of lines which bisect the vertical angles formed by the two intersecting lines. What would the locus be in space?

2. Show that the theorem of Art. 195 is really a special case of Ex. 1.

3. Draw an obtuse triangle and the perpendicular bisector of each side. What is the test for the accuracy of the construction?

4. Draw a triangle in which one angle is a right angle, and construct the perpendicular bisector of each side.

5. Draw an acute triangle and construct the perpendicular bisector of each side. What is the test for the accuracy of the construction in this case?

196. Concurrent lines. If three or more lines intersect at one point, they are called *concurrent lines.* Several important locus problems deal with concurrent lines.

197. Theorem. *The perpendicular bisectors of the sides of a triangle meet (are concurrent) in a point which is equidistant from the three vertices of the triangle.*

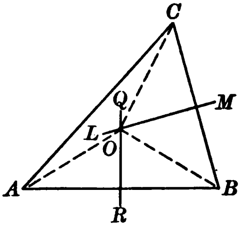


FIG. 218

Given the triangle *ABC*.

To prove that *the perpendicular bisectors of the sides of the triangle are concurrent in a point equidistant from the three vertices of the triangle.*

Proof

STATEMENTS	REASONS
1. Let QR and LM be the perpendicular bisectors of AB and BC respectively. They must meet in some point O.	1. Obviously, two lines perpendicular to two intersecting lines must meet.
2. Join O to A, B, and C.	2. One and only one straight line can be drawn between two points.
3. Then $OA = OB = OC$.	3. Any point on the perpendicular bisector of a line segment is equidistant from the ends of the line.
4. $\therefore OA = OC$.	4. Things equal to the same thing are equal to each other.
5. ∴ O is equidistant from A and C and must, therefore, lie on the perpendicular bisector of AC.	5. The locus of all points equidistant from the ends of a line segment is the perpendicular bisector of the segment.
6. ∴ the perpendicular bisectors of a triangle are concurrent.	6. By definition of concurrent lines.

NOTE. The point of intersection of the perpendicular bisectors of the sides of a triangle is called the *circumcenter* of the triangle.

198. Concentric circles. Circles which have the same center are called *concentric circles.*

EXERCISES

1. Show by the definition of a circle given in Art. 190 that all radii of the same circle or of congruent circles are equal.

2. Prove by superposition that two circles are congruent if they have equal radii.

3. All diameters of the same circle or of congruent circles are equal.

4. A diameter of a circle bisects the circle (use superposition).

5. Any chord that divides a circle into two equal parts is a diameter of the circle.

6. A point lies within, on, or outside a circle, according as the distance from the center is less than, equal to, or greater than a radius.

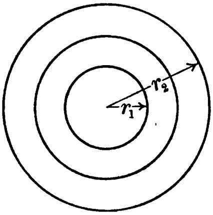


FIG. 219

7. In the same circle or in equal circles radii that form unequal central angles intercept unequal arcs on the circle.

8. State and prove the converse of Ex. 7.

9. What is the locus of a point which is equidistant from two concentric circles (see Fig. 219)?

10. What is the locus in a plane of a point P at a distance d from a given circle?

199. Inscribed polygon; circumscribed circle. A polygon is *inscribed in a circle* if all its vertices are on the circle, as in the polygon $ABCDE$ in Fig. 220. The circle is said to be *circumscribed about the polygon*, and its center is called the *circumcenter* of the polygon.

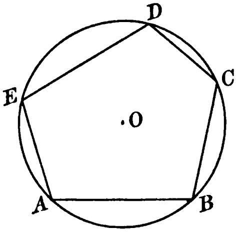


FIG. 220

200. ***Construction problem.*** *To circumscribe a circle about a given triangle.*

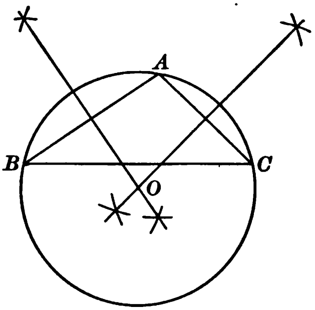


FIG. 221

Given the triangle *ABC*.

Required *to circumscribe a circle about the* $\triangle ABC$.

Construction. This problem depends upon the theorem of Art. 197. Draw the perpendicular bisectors of AC and AB; they will intersect at some point O, as shown in Art. 197. From O as a center and with a radius equal to OA describe the circle ABC, as shown.

Then ABC is the required circle.

Proof. The proof is left to the student.

201. ***Corollary 1.*** *Three points which are not collinear determine a circle.*

202. ***Corollary 2.*** *A circle cannot be drawn through three collinear points.*

203. ***Corollary 3.*** *A straight line cannot cut a circle in more than two points.*

204. ***Corollary 4.*** *Two circles cannot intersect in more than two points.*

EXERCISES

1. To find the center of a given circle.
2. To bisect a given arc.
3. Given an arc of a circle, to find the center of the circle.
4. Give a practical application of the construction in Ex. 3.

205. Theorem. *The bisectors of the three angles of any triangle are concurrent in a point equidistant from the three sides.*

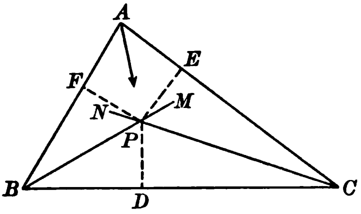


FIG. 222

Given the triangle *ABC*.

To prove that *the bisectors of the three angles of the* $\triangle ABC$ *are concurrent in a point equidistant from the three sides of the triangle.*

Proof

STATEMENTS	REASONS
1. Draw the bisectors BM and CN of the ∠s B and C respectively. They will meet at some point P.	1. Obviously, they cannot be parallel if they are drawn within two lines which intersect.
2. Draw PD, PF, and $PE \perp BC$, BA, and AC respectively.	2. From a point outside a given line one and only one perpendicular can be drawn to the line.
3. Then $PF = PD = PE$.	3. All points on the bisector of an angle are equidistant from the sides of the angle.
4. $\therefore PF = PE$.	4. Things equal to the same thing are equal to each other.
5. $\therefore P$ must lie on the bisector of the $\angle A$.	5. The locus of all points equidistant from the sides of an angle is the bisector of the angle.
6. $\therefore$ the bisectors of the angles of the $\triangle ABC$ are concurrent.	6. By definition of concurrent lines.

NOTE. The point of intersection of the bisectors of the three angles of a triangle is called the *incenter* of the triangle.

EXERCISE

To inscribe a circle in a given triangle.

206. Theorem. *The three altitudes of a triangle are concurrent.*

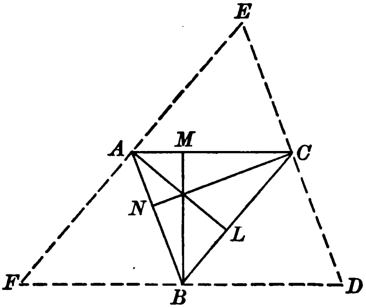


FIG. 223

Given the triangle *ABC* with the three altitudes *AL*, *CN*, and *BM*.

***To prove that** AL, CN, and BM are concurrent.*

Proof

STATEMENTS	REASONS
1. Through A, B, and C draw EF, FD, and $DE \parallel BC$, CA, and AB respectively.	1. Through a point outside a given line one and only one line can be drawn parallel to that line.
2. Then $FBCA$ is a parallelogram.	2. By definition.
3. $\therefore FB = AC$.	3. Opposite sides of a parallelogram are equal.
4. In like manner, $BDCA$ is a parallelogram, and $BD = AC$.	4. For the same reasons.
5. $\therefore FB = BD$.	5. Things equal to the same thing are equal to each other.

Proof (*Continued*)

STATEMENTS	REASONS
6. But $BM \perp AC$.	6. By definition of an altitude.
7. $\therefore BM \perp FD$.	7. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other.
8. $\therefore$ BM is the perpendicular bisector of FD.	8. By 5 and 7 above.
9. In like manner, AL is the perpendicular bisector of EF, and CN is the perpendicular bisector of ED.	9. For the same reasons.
10. $\therefore$ AL, BM, and CN are concurrent.	10. The perpendicular bisectors of the sides of a triangle are concurrent.

NOTE. The point of intersection of the three altitudes of a triangle is called the *orthocenter* of the triangle.

EXERCISES

1. Draw four isosceles triangles on the same base and connect their vertices. What geometric truth is illustrated by the figure you have drawn?

2. The bisectors of two vertical angles are collinear.

3. The bisector of one vertical angle bisects the other also.

4. What is the locus of the vertices of all triangles having the same given base and equal medians to this base?

5. Find a point equidistant from the sides AB, BC, and CD of a parallelogram $ABCD$.

6. Does the orthocenter of a triangle necessarily fall inside the triangle?

7. The perpendicular bisectors of the four sides of an isosceles trapezoid are concurrent.

8. If P is the orthocenter of the $\triangle ABC$, show that A is the orthocenter of the $\triangle BCP$, B is the orthocenter of the $\triangle ACP$, and C is the orthocenter of the $\triangle ABP$.

207. Theorem. *The medians of a triangle are concurrent in a point which is two thirds of the distance from a vertex to the midpoint of the opposite side.*

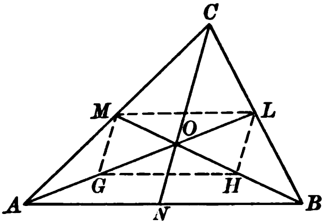


FIG. 224

Given the triangle *ABC* and the medians *AL*, *BM*, and *CN*.

To prove that *AL, BM, and CN are concurrent in a point two thirds of the distance from each vertex to the midpoint of the opposite side.*

Proof

STATEMENTS	REASONS
1. AL and BM must intersect at some point O.	1. Obviously, lines drawn from the vertices of two angles of a triangle and lying within these angles must intersect.
2. Bisect AO and BO at G and H respectively.	2. A line may be bisected.
3. Draw GH, HL, LM, and MG.	3. One and only one straight line can be drawn between two points.
4. Then $GH \parallel AB$, and $GH = \frac{1}{2}AB$.	4. If a line is drawn from the midpoint of one side of a triangle to the midpoint of another side, the line is parallel to the third side and equal to one half of it.
5. $ML \parallel AB$, and $ML = \frac{1}{2}AB$.	5. For the same reason.
6. $\therefore GH \parallel AB$, and $GH = ML$.	6. Because they are each parallel to AB and equal to $\frac{1}{2}AB$.

Proof (*Continued*)

STATEMENTS	REASONS
7. $\therefore GHLM$ is a parallelogram.	7. If a pair of opposite sides are equal and parallel, the quadrilateral is a parallelogram.
8. Whence $MO = OH$, and $GO = OL$.	8. The diagonals of a parallelogram bisect each other.
9. Then $MO = OH = HB$, and $LO = OG = GA$.	9. By 2 and 8 above.
10. $\therefore AO = 2\,OL$, or $AO = \frac{2}{3}AL$, and $BO = \frac{2}{3}BM$.	10. Obvious from what is shown above.
11. In like manner, AL and CN intersect in a point which is two thirds of the distance from each vertex to the midpoint of the opposite side. In other words, CN also passes through O.	11. For the same reasons as given above.
12. $\therefore$ the medians of a triangle are concurrent in a point two thirds of the distance from each vertex to the midpoint of the opposite side.	

208. Center of gravity of a triangle; centroid. The point of intersection of the medians (Fig. 224) is called the *center of gravity*, or *centroid*, of the triangle. This point is a trisection point of each median. If a cardboard triangle is supported at the center of gravity on a sharp point, it will be found that the triangle balances. The notion of the center of gravity is very useful in the study of mechanics.

EXERCISES

1. Cut out a triangle from stiff cardboard, construct the three medians carefully, and see if the triangle balances when it is supported at the centroid by a pointed object.

2. The bisector of an interior angle and the bisectors of the two nonadjacent exterior angles of a triangle are concurrent.

NOTE. The point of intersection of the bisector of an interior angle and those of the two nonadjacent exterior angles of a triangle is called an *excenter* of the triangle.

3. How many excenters does a triangle have?

4. Using the excenters of a given triangle as centers, construct three circles tangent to the three sides of the triangle.

NOTE. A circle is tangent to a line if it touches it at only one point. The three circles thus constructed are called *escribed circles.*

5. How may a fourth circle tangent to the sides of the triangle in Ex. 4 be drawn?

6. The perpendicular bisectors of the legs of a right triangle intersect on the hypotenuse.

7. Where do the altitudes of a right triangle intersect? Prove your answer.

8. The perpendicular bisectors of the four sides of an isosceles trapezoid are concurrent.

9. The bisectors of two exterior angles of a triangle meet at an angle which is equal to one half the third exterior angle.

10. If the bisectors of the angles of a quadrilateral are concurrent, the sum of two opposite sides equals the sum of the other two opposite sides.

11. Prove the theorem of Art. 207 by using Fig. 225.

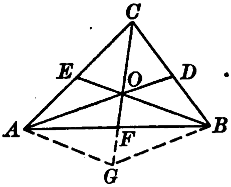


FIG. 225

SUGGESTION. Show that the medians AD and BE meet at O. Draw CG through O and draw $AG \parallel EB$. Draw GB and prove that $AGBO$ is a parallelogram. Then $AF = FB$.

209. Intersection of loci. It often happens that a point fulfills two conditions, each of which involves a separate locus. In this case we must try to find the points common to two loci. Sometimes there is no solution. We usually try to discuss all the possibilities or show that the solution is impossible.

For example, suppose that we wish to find the locus of a point which is equidistant from two given points and at a given distance from a third point.

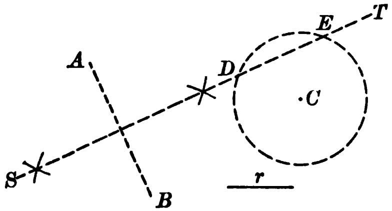


FIG. 226

Solution. Let A and B (Fig. 226) be the two given points, C the third given point, and r the given distance.

Since the point is equidistant from both A and B it lies somewhere on the perpendicular bisector ST of the line AB. Since the point is at the distance r from C, it lies also on a circle of radius r and center C. The line ST cuts the circle in the points D and E.

Hence the required locus is D and E.

There is no solution if r is not long enough to give a circle large enough to intersect ST. If the circle is tangent to ST, there is only one solution.

EXERCISES

1. What is the locus of a point a inches from a given point and b inches from a given line? Discuss fully.

2. Locate a point which lies in a given line and which is at the same time equidistant from two given points. Discuss fully.

3. What is the locus of a point equidistant from the sides of a given angle and at a given distance from a given point? Discuss fully.

4. Find a point equidistant from two given points and also equidistant from the sides of a given angle. Discuss fully.

5. Locate a point on a given circle and equidistant from the ends of a given line.

6. Construct a circle which shall pass through two given points and which shall have its center on a given line.

7. A miser buried some gold 10 ft. from his front-yard fence and equidistant from two trees in the yard. Show how the spot where the treasure is buried can be located.

8. Locate a point which is equidistant from two given points and also equidistant from two given parallel lines.

9. Locate a point which shall lie on one side of a given triangle and which shall be equidistant from the other two sides.

10. To find a point on a given line which shall be equidistant from two given points not on the line.

11. To construct a triangle having given the sides a and c and the $\angle A$.

210. Uses of directed line segments. In geometry we may think of a line segment as having either of two directions, positive or negative, just as in our algebra work we spoke about $+8$ and -8. If on a given line we take a point O as origin and say that distances from O in one direction along the line shall be positive (+) and those in the opposite direction shall be negative (−), we can locate points on the line if we know their distance and direction from the origin O. Thus, if on a horizontal line distances to the right of O are positive and those to the left of O are negative, and it is given that a segment OP is $+3$ cm. in length, we can locate the point P as the point on the line 3 cm. to the right of O. If the segment OP' is -3 cm. in length, the point P' is 3 cm. to the left of O. Or, if we have given on the line the point P'' 5 cm. to the right of O, we see that the segment OP'' is $+5$ cm. in length.

If the student will recall the ordinary thermometer scale, where we read 8° above zero as "$+8°$" and 8° below zero as "$-8°$," he will see that this is only a common use of a directed line segment (see Art. 167).

If a directed line segment lies on a directed line, its direction may be either the same or opposite to that of the line upon which it lies. Thus, if a directed line segment on a line x is 8 units long and has the same direction as the line x, we should call the segment $+8$; but if it has the opposite direction, we should call it -8. That is, we must call a directed line segment which lies on a directed line positive or negative according as it has the same direction as or the opposite direction to the directed line.

211. Terms used in graphical representation. Certain terms which are used in mathematics in connection with graphical representation will now be given and illustrated. The plane of the squared paper (Fig. 227) is divided into four quarters by the lines XX' and YY' at right angles to each other, and these quarters are called *quadrants*. They are usually numbered I, II, III, and IV, I being the quadrant above XX' and to the right of YY', II the one above XX' and to the left of YY', and so on in counterclockwise order. The lines XX' and YY' are called *axes*, or *coördinate axes*, XX' being the *horizontal axis* and YY' the *vertical axis*. The point O, the intersection of XX' and YY', is called the *origin*. The scale used is indicated on the axes.

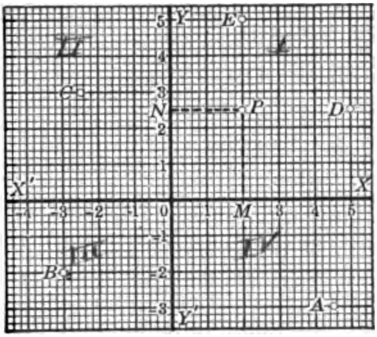


FIG. 227

To locate P, any point on the squared paper, perpendiculars are drawn to the axes. The line PM is called the *ordinate of P*, and the line PN is called the *abscissa of P*; together they are called the *coördinates of P*. The distances on OX and OY measured from O are positive; those on OX' and on OY' are negative.

In Fig. 227 the abscissa of the point P is $+2$ and the ordinate is $+2\frac{1}{2}$; and to describe the point P we may write "the point $(2, 2\frac{1}{2})$," "$P(2, 2\frac{1}{2})$," or simply "$(2, 2\frac{1}{2})$," in which case the words "the point" are understood. The student should notice that the abscissa is written first and the ordinate second. The process of locating on squared paper a point which corresponds to a given pair of coördinates is called *plotting the point.*

EXERCISES

1. In Fig. 227 what is the abscissa of the point A? the point B? the point C? the point D? the point E?

2. In Fig. 227 what is the ordinate of the point A? the point B? the point C? the point D? the point E?

3. Give the coördinates of the points A, B, C, D, and E in Fig. 227.

4. On a sheet of graph paper draw a pair of coördinate axes which intersect near the center of the paper, and plot the following points: $(3, 4)$, $(5, 6)$, $(4, -5)$, $(-3, 8)$, $(-5, -3)$, $(-4\frac{1}{3}, -2\frac{1}{4})$.

5. On a sheet of graph paper locate the points $A(3, 2)$, $B(4, 3)$, $C(3, 8)$, and $D(6, 7)$. What kind of figure is formed when the points A, B, C, and D are connected? Draw the diagonals of the figure and state the coördinates of the point of intersection of the diagonals.

6. What is the locus of all points which are 2 units to the right of the y-axis? which are 4 units to the left of the y-axis? Write the equation of each line.

7. What is the locus of all points which are 3 units above the x-axis? which are 5 units below the x-axis? Write the equation of each line.

8. The distances x and y from the coördinate axes to a point satisfy the equation $y = 3x + 2$. Find by means of a graph the locus of the point.

9. The sum of the distances of a point from the coördinate axes is 8. What is the locus of the point? Illustrate by means of a graph.

10. The difference between the distances of a point from two perpendicular lines is 5. What is the locus of the point?

11. What is the locus of a point whose ordinate is 4 units greater than its abscissa?

12. A point is three times as far from the x-axis as it is from the y-axis. Draw the locus of the point.

13. Draw the following loci:

(a) $y = 4x - 3$. (c) $2x + y = 5$. (e) $3x + 4y = 8$.
(b) $x = 3y - 1$. (d) $x - 3y = 6$. (f) $4x - 5y = 2$.

14. Draw the following loci:

(a) $3x = y^2$. (b) $x^2 = 5x$. (c) $xy = 2$.

212. *Problem.* A very common locus is shown by the following problem, which the student should solve carefully: *Find the locus of a point which moves so that its distance from a fixed point F (Fig. 228) is always equal to its distance from a fixed line RS.*

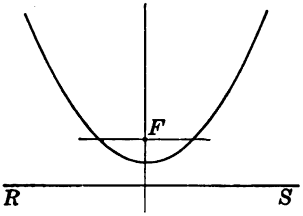


FIG. 228

SUGGESTION. Take the point F 2 in. from the line RS. Then proceed to find a number of points which are equidistant from F and RS, some below F and some above F, both to the right and left. Draw a smooth curve through the points.

The curve constructed in Fig. 228 is called a *parabola.* The student should remember that the graph of a quadratic equation in one variable is a parabola, and that one method of solving a quadratic equation is the graphic method. We have learned to solve quadratic equations by the following methods:

1. The graphic method.
2. Factoring.
3. The method of completing the square.

The student should also remember the following facts about solving quadratic equations:

1. The graphic method of solving a quadratic equation gives a good picture of what the solution means, but in a great many cases it gives only approximate solutions.
2. The method of factoring is easy and is almost always more accurate than the graphic method, but the method fails when the trinomial cannot be factored.
3. The method of completing the square always gives the correct roots, but the process is long and often difficult. It is, however, the fundamental method and should be learned.

213. Solution of quadratic equations by formula. From the method of solving a quadratic equation by completing the square, we can obtain a formula the use of which gives us a much more powerful method than any other which we have considered. All quadratic equations may be written in the form $ax^2 + bx + c = 0$; that is, when the given equation is written in its simplest form we shall have a certain number of x^2's, a certain number of x's, and some constant term (arithmetic number).

If we solve the equation $ax^2 + bx + c = 0$ by the method of completing the square, we obtain a formula for the two roots of a quadratic equation by which we can solve any given quadratic equation by merely substituting the values of a, b, and c (the constants) of the given equation in the formula.

We shall now proceed to develop the formula and illustrate its use.

For example, solve, by completing the square, the general equation

$$ax^2 + bx + c = 0.$$

Solution. Subtracting c from both members, we have

$$ax^2 + bx = -c.$$

Dividing both members by a, we have

$$x^2 + \frac{bx}{a} = \frac{-c}{a}.$$

Adding $\frac{b^2}{4a^2}$ to both members, we have

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both members, we have

$$x + \frac{b}{a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a},$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence the formulas for the two roots of any quadratic equation are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The student should thoroughly memorize these formulas in order to insure absolute accuracy in solving quadratic equations. We shall now show how the formulas are used.

For example, solve, by using the quadratic formula, the equation

$$2x^2 + 5x + 2 = 0.$$

Solution. In solving the equation we need to remember that in the formula, a represents the coefficient of x^2, b represents the coefficient of x, and c represents the constant (arithmetic) term. Consequently, in the given equation, $a = 2$, $b = 5$, and $c = 2$.

Substituting these values in the formulas, we have

$$x_1 = \frac{-5 + \sqrt{25 - 16}}{4} = -\frac{1}{2}$$

and

$$x_2 = \frac{-5 - \sqrt{25 - 16}}{4} = -2.$$

Hence the roots are $-\frac{1}{2}$ and -2, as can be verified by substituting these results in the given equation.

EXERCISES

Solve the following equations by using the quadratic formula:

1. $6x^2 - 11x + 5 = 0.$
2. $2x^2 - x - 6 = 0.$
3. $2x^2 + x - 15 = 0.$
4. $x^2 - 9x = 36.$

HINT. Subtract 36 from both sides of the equation, and then apply the formula.

5. $x^2 + 15x = -44.$
6. $x^2 - 72 = 6x.$

HINT. Subtract $6x$ from both sides of the equation, and then apply the formula.

7. $3x^2 = 6 - 7x.$
8. $6 + 11y = -18y^2 - 20.$
9. $14a^2 + 2a = 28a - 10a^2 + 5.$
10. $11a^2 - 10a = 24 - 10a^2.$
11. $6b^2 - 13b = 10b - 21.$
12. $23p + 20 = 10p^2 - 23.$
13. $\frac{8}{x} - 2x - \frac{2}{3} = 0.$
14. $\frac{3}{5 - x} = 2x.$

APPLIED PROBLEMS

Solve by any method the quadratic equations involved in the following problems:

1. In Fig. 229 AB is the diameter of the circle ACB, $DB = 15$, and $AC = 4$. What is the length of AD?

2. In Fig. 229, if $BC = 7$ and $AD = 3$, what is the length of AB?

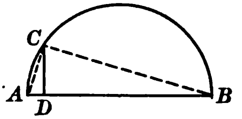


FIG. 229

3. In two complementary angles the square of the number of degrees in the larger exceeds by 500 the product of the number of degrees in one angle by the number of degrees in the other angle. Find the number of degrees in each angle.

4. In two supplementary angles the product of the two numbers which represent the number of degrees in each angle is 7875. Find the size of each angle.

5. A man sold a colt for $96, thereby gaining a per cent which was equal to the cost of the colt in dollars. Find the cost of the colt.

6. A needs 4 da. more than B to cultivate a field of corn. They can do the work together in 3 da. In how many days can B do the work alone?

7. In a number of two digits the tens' digit exceeds the units' digit by 3. If the number is divided by the product of its digits, the quotient is $\frac{26}{5}$. Find the number.

8. In going a distance of 36 mi. on a motorcycle, a man figured that he would have arrived 15 min. earlier if he had traveled 2 mi. per hour faster. Find the rate of speed at which he actually traveled.

214. The circle as the graph of a quadratic equation. In the equation $x^2 + y^2 = 25$, if we assign values to y and compute the corresponding values of x so as to complete

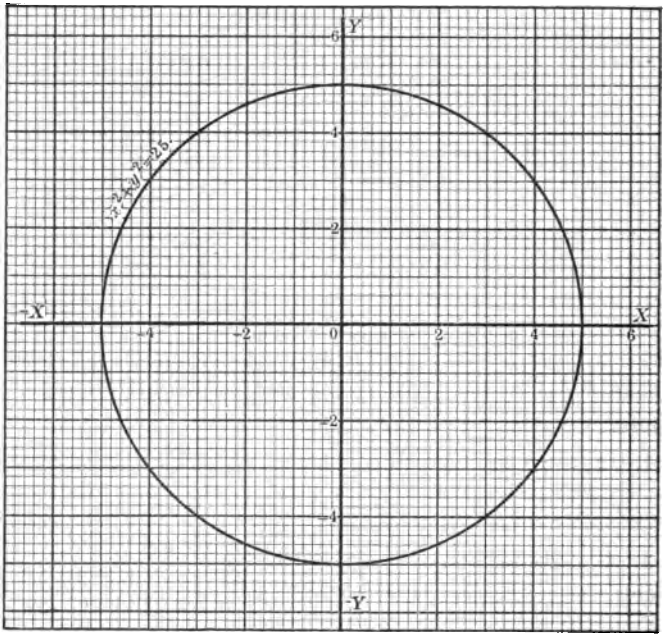


FIG. 230

the table below, we obtain the coördinates for the points of the curve in Fig. 230, which is obviously a circle. If we consider the x and y values of any point on the curve, then, by the theorem of Pythagoras,[1] $x^2 + y^2 = 25$. Similarly, we can show that the graph of the equation $x^2 + y^2 = r^2$ is always a circle whose radius is r and whose center is at the origin; that is, at the point (0, 0).

x	y
0	± 5
2	
3	± 4
	0
	± 3

---

[1] In any right triangle the square of the hypotenuse equals the sum of the squares of the other two sides. (We are assuming that this theorem is known to all by this time.)

EXERCISES

Construct the graph for each of the following equations:

1. $x^2 + y^2 = 16.$

2. $x^2 + y^2 = 36.$

3. $x^2 + y^2 = 40.$

4. $x^2 + y^2 = 12.$

215. Simultaneous equations. In Fig. 231 the graphs of the equations $x^2 + y^2 = 25$ and $3x + 4y = 20$ have been drawn to the same scale with the same axes. In how many places does the line intersect the circle? Inspect the graph carefully and determine the values of x and y

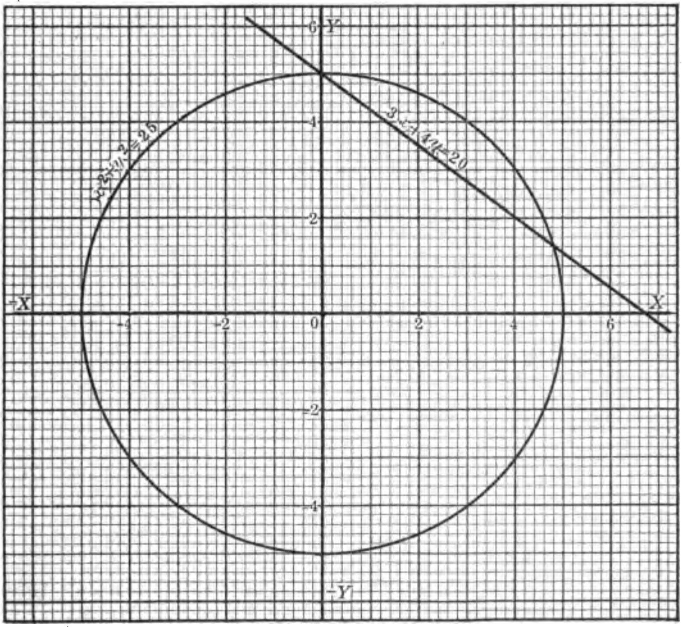


FIG. 231

of the two points that lie on both the line and the circle. Substitute the values found and see if they satisfy the values of both equations. What difficulties do you meet in determining the values of x and y for one point?

The foregoing discussion suggests that it is sometimes difficult to determine the exact point of intersection by the graphic method. Fortunately this difficulty may be met by the following algebraic method.

Solution. Solving the linear equation $3x + 4y = 20$ for x, we obtain

$$x = \frac{20 - 4y}{3}.$$

Substituting this value for x in the quadratic equation $x^2 + y^2 = 25$, we obtain

$$\left(\frac{20 - 4y}{3}\right)^2 + y^2 = 25,$$

or

$$25y^2 - 160y + 175 = 0.$$

Solving this equation for y, we have $y = 5$, or 1.4.

Substituting the values found for y in the linear equation

$$x = \frac{20 - 4y}{3},$$

we obtain

$$x = 0, \text{ or } 4.8.$$

Hence the points of intersection of the line and circle are $(0, 5)$ and $(4.8, 1.4)$.

EXERCISES

In each of the following pairs determine the points of intersection by a graph and verify by the algebraic method:

1. $\begin{cases} x^2 + y^2 = 25, \\ 3x + 4y = 26. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 16, \\ x + y = 4. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 36, \\ 4x + y = 6. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 12, \\ x - y = 6. \end{cases}$

216. *Problem.* The following problem, which the student should solve carefully, introduces another interesting locus: *Find the locus of a point the sum of whose distances from two fixed points is always constant.*

FP	1	$1\frac{1}{8}$	$1\frac{1}{2}$	2	$2\frac{1}{2}$	$2\frac{7}{8}$	3
$F'P$	3	$2\frac{7}{8}$	$2\frac{1}{2}$	2	$1\frac{1}{2}$	$1\frac{1}{8}$	1

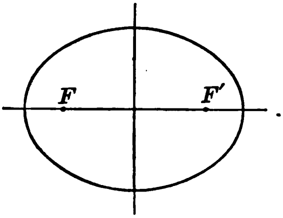


FIG. 232

SUGGESTION. Take two points, F and F', say 2 in. apart. Let the point P move so that $FP + F'P = 4$ in. Then the above table will give some of the possible pairs of values from which the locus may be drawn.

Plot all the points in the preceding table by means of intersecting arcs, and draw a smooth curve through the points.

Such a curve (see Fig. 232) is called an *ellipse.*

217. The ellipse as the graph of a quadratic equation. In order to draw the graph of the equation

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

we proceed as follows:

1. Solve for y, obtaining

$$y = \pm \tfrac{4}{5}\sqrt{25 - x^2}.$$

x	y
0	± 4
± 1	± 3.9
± 2	± 3.7
± 3	
± 4	
± 5	

2. Assign values to x and obtain the corresponding values of y, as in the accompanying table.

3. Plot the points whose coördinates are given in the table and draw a smooth curve through these points, as in Fig. 233.

The graph of every equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (or the form $b^2x^2+a^2y^2=a^2b^2$) is an ellipse.

The ellipse is of particular interest in astronomy, since the paths of heavenly bodies are approximate ellipses.

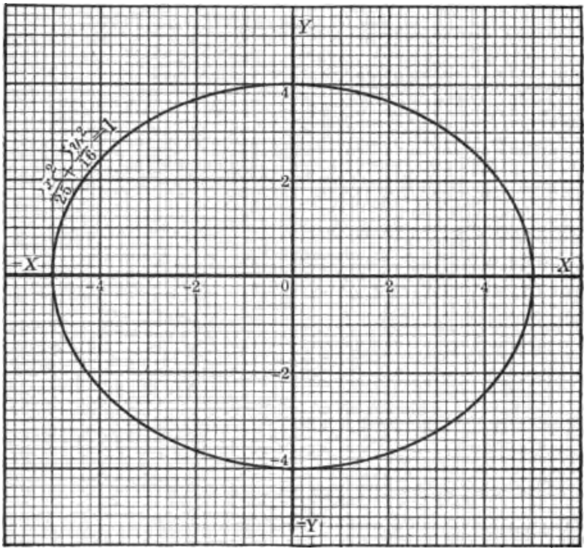


FIG. 233

EXERCISES

Draw the graph of each of the following pairs of equations, determine whether their graphs intersect, write the coördinates of the intersection points, and check by solving algebraically:

1. $\begin{cases} \frac{x^2}{16}+\frac{y^2}{9}=1, \\ 3x+4y=12. \end{cases}$

2. $\begin{cases} x^2+4y^2=25, \\ 2x-y=4. \end{cases}$

3. $\begin{cases} 2x^2+3y^2=11, \\ y-3x=7. \end{cases}$

4. $\begin{cases} \frac{x^2}{25}+\frac{y^2}{9}=1, \\ 2x-y=14. \end{cases}$

218. Simultaneous quadratic equations. Fig. 234 shows the graphic solution of the system

$$\begin{cases} x^2 + y^2 = 25, & (1) \\ 4x^2 + 9y^2 = 144. & (2) \end{cases}$$

From the preceding pages we know that the graph of equation (1) is a circle and that the graph of equation (2) is an ellipse. In Fig. 234 these curves have been plotted to the same scale on the same axes. The two graphs intersect in four points, which means that there are four values of x and four corresponding values of y which satisfy both equations. See if you can pick out the correct values.

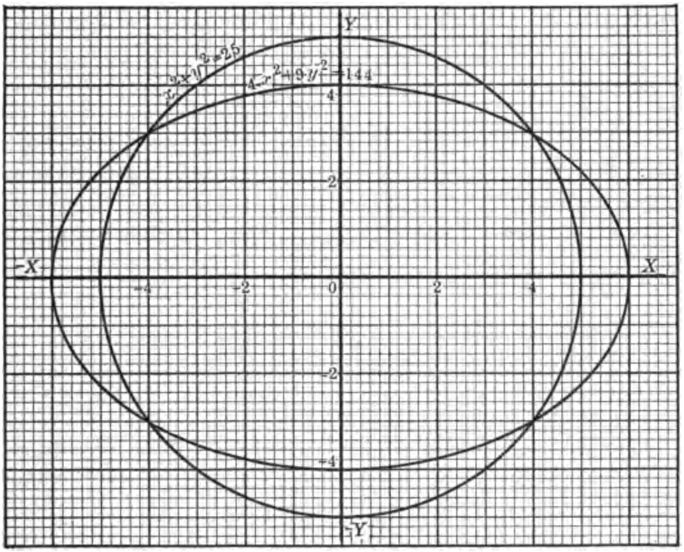


FIG. 234

Algebraic solution. The given equations are

$$x^2 + y^2 = 25 \tag{1}$$

and

$$4x^2 + 9y^2 = 144. \tag{2}$$

If we multiply equation (1) by 4 and leave equation (2) as it is, we have

$$4\,x^2 + 4\,y^2 = 100, \tag{3}$$

and

$$4\,x^2 + 9\,y^2 = 144. \tag{4}$$

Subtracting (3) from (4), we have

$$5\,y^2 = 44,$$

$$y^2 = \tfrac{44}{5},$$

and

$$y = \pm\sqrt{8.8}.$$

Substituting $y = \sqrt{8.8}$ in equation (1), we have

$$x^2 + 8.8 = 25,$$

$$x^2 = 16.2,$$

and

$$x = \pm\sqrt{16.2}.$$

We therefore find that two of the points of intersection are $(\sqrt{16.2}, \sqrt{8.8})$ and $(-\sqrt{16.2}, \sqrt{8.8})$.

In like manner, by substituting $y = -\sqrt{8.8}$ in equation (1), we have

$$y^2 + 8.8 = 25,$$

$$y^2 = 16.2,$$

and

$$y = \pm\sqrt{16.2}.$$

We thus find that the two other points of intersection are $(\sqrt{16.2}, -\sqrt{8.8})$ and $(-\sqrt{16.2}, -\sqrt{8.8})$.

These square roots may now be found from a table of roots, and the results used to check the answers obtained by the graphic method.

EXERCISES

Solve each of the following systems graphically and check by an algebraic solution:

1. $\left\{\begin{array}{l} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 36. \end{array}\right.$

2. $\left\{\begin{array}{l} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 30. \end{array}\right.$

3. $\left\{\begin{array}{l} \dfrac{x^2}{36} + \dfrac{y^2}{25} = 1, \\ x^2 + y^2 = 25. \end{array}\right.$

4. $\left\{\begin{array}{l} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 3. \end{array}\right.$

219. Mechanical construction of an ellipse. The ellipse in Fig. 232 might have been constructed by the following mechanical means:

Drive two pins firmly into your paper 2 in. apart. Make a small loop in the end of a piece of string or strong thread about 6 in. long, and place the loop over one of the pins. Adjust the string around the other pin so that there shall be exactly 4 in. of string between the pins. Place a pencil point against the string, and keeping the string taut, move the point of the pencil around the pins until a complete revolution is made. The pencil point will trace the ellipse.

220. *Problem.* Another interesting locus is illustrated in the following problem, which the student should solve carefully: *Find the locus of a point the difference of whose distances from two fixed points is always constant.*

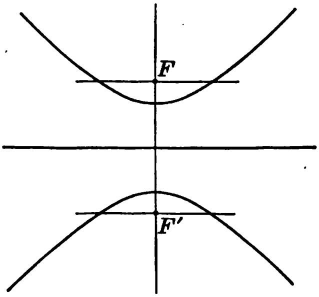


FIG. 235

SUGGESTION. On your paper take the two fixed points F and F' 3 in. apart, and let the constant difference be 2 in. Make a table similar to the one in Art. 216, being careful to take $FP > F'P$ for some of the points and $F'P > FP$ for others. Locate the points by means of intersecting arcs, and draw a smooth curve through them.

Such a curve (see Fig. 235) is called a *hyperbola.*

221. The hyperbola as the graph of a quadratic equation. In order to draw the graph of the equation $\frac{x^2}{25} - \frac{y^2}{16} = 1$ we write the equation in the form $y = 0.8\sqrt{x^2 - 25}$; that is, we solve the equation for y in terms of x. If we then assign values to x and compute the corresponding values of y to complete the following table, we can obtain the points necessary for the graph:

x	± 5	$\pm \frac{25}{4}$	± 7	± 8	± 10
y	0	± 3	± 3.9		

If we plot the points whose coördinates are given by the table and draw a smooth curve through them, we obtain the graph of Fig. 236.

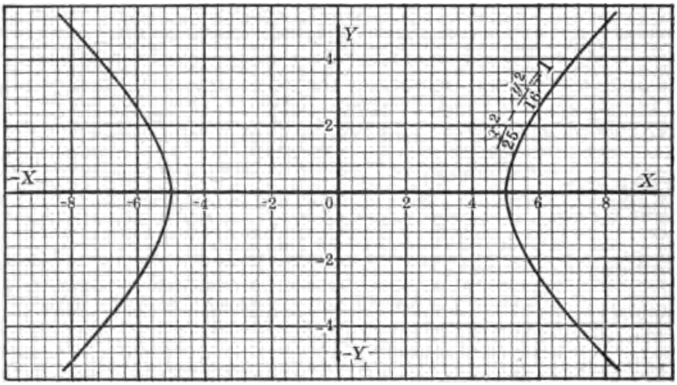


FIG. 236

The parabola, ellipse, and hyperbola are called *conic sections.* Since we may consider the circle as a special form of the ellipse, the circle is also called a conic section.

EXERCISES

Solve each of the following systems by the graphic method:

1. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 16. \end{cases}$ 2. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 25. \end{cases}$ 3. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 36. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 4. \end{cases}$ 6. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1. \end{cases}$ 7. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1. \end{cases}$

NOTE. In order to save time Exs. 1–4 may be plotted on the same sheet with the same coördinate axes, but if this is done it will be advisable to use different colored pencils or ink. Exs. 5–7 constitute a similar group.

8. $\begin{cases} x^2 + y^2 = 25, \\ xy = 12. \end{cases}$ 10. $\begin{cases} x^2 - y^2 = -16, \\ x - 3y = 12. \end{cases}$

9. $\begin{cases} xy = 6, \\ 3x + y = 9. \end{cases}$ 11. $\begin{cases} x = y - 1, \\ 5x^2 + 2y^2 = 51. \end{cases}$

222. Review. The following exercises serve as an oral review of simultaneous quadratic equations.

EXERCISES

1. In how many points can a straight line intersect a circle? a parabola? an ellipse? a hyperbola? (See Fig. 237.)

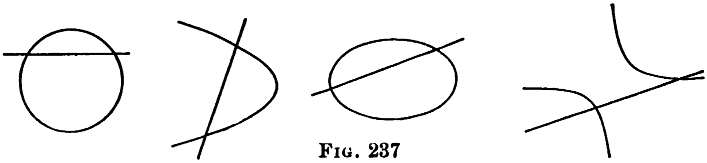


FIG. 237

2. In each of the following systems state the number of values which x and y have:

(a) $\begin{cases} 2x + y = 6, \\ x^2 + y^2 = 25. \end{cases}$

(b) $\begin{cases} x + 3y = 2, \\ x^2 + 4y^2 = 8. \end{cases}$

(c) $\begin{cases} y^2 = 8x, \\ y = 6x. \end{cases}$

(d) $\begin{cases} \dfrac{x^2}{36} - \dfrac{y^2}{25} = 1, \\ x = y. \end{cases}$

3. In how many points can an ellipse intersect a circle? a parabola? an ellipse? a hyperbola? (See Fig. 238.)

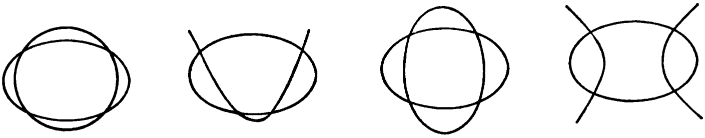


FIG. 238

4. State the number of roots in each of the following systems:

(a) $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1, \\ x^2 + y^2 = 8. \end{cases}$

(b) $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1, \\ y^2 = 8x. \end{cases}$

(c) $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1, \\ \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1. \end{cases}$

(d) $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1, \\ xy = 2. \end{cases}$

223. Formulas involving quadratic terms. The chapter will close with a drill in solving and evaluating formulas which involve quadratic factors.

EXERCISES

1. The law of falling bodies is represented by the formula $h = \dfrac{v^2}{2g}$.

(a) Solve for v.

(b) Find v (velocity) if h (height) $= 100$ ft. and g (acceleration of gravity) $= 32.2$ ft. per second.

2. The pressure of wind blowing perpendicular to a plane surface is given by the formula $P = \frac{AV^2}{256}$.

(a) Solve for V.

(b) What is P (pressure in pounds per square inch) if A (area) $= 1$ sq. in. and V (velocity of wind) $= 32$ mi. per hour?

3. The volume of a cone is given by the formula $V = \frac{h}{3}\pi r^2$.

(a) Solve for r.

(b) Find r (radius) if V (volume) $= 16$ cu. in. and h (height) $= 3\frac{2}{3}$ in. (Use $\pi = \frac{22}{7}$.)

4. The formula for the area of a circle is $A = \pi r^2$.

(a) Solve for r.

(b) Find r (radius) when A (area) $= 154$ sq. in.

5. The elevation of the outside rail around a curve on a railroad is given by the formula $E = \frac{4\,dv^2}{5\,r}$.

(a) Solve for v.

(b) For a train traveling around a curve what should be the maximum value of v (velocity in miles per hour) if d (distance between centers of the rails) $= 4\frac{3}{4}$ ft., r (radius of the curve) $= 1750$ ft., and E (elevation of outside rail in feet) $= 1\frac{1}{4}$ in.?

6. The minimum number of bolts that are required for safety around the head of the cylinder of a steam engine is given by the formula $N = \frac{p}{2400}\left(\frac{D}{d}\right)^2$.

(a) Solve for p.

(b) Calculate p (the pitch, or distance between centers of the bolts) if N (number of bolts) $= 36$, D (diameter of the cylinder in inches) $= 36$, and d (diameter of the bolts in inches) $= 1\frac{1}{4}$.

7. The formula for the volume of a cylinder is $V = \pi r^2 h$.

(a) Solve for r.

(b) Find r (radius) if V (volume) $= 120$ cu. in. and h (height) $= 10$ in.

8. The area of a sphere is given by the formula $A = 4\pi r^2$.

(a) Solve for r.

(b) Find r (radius) when A (area) = 400 sq. in.

9. The distance formula is $d = rt$.

(a) Solve for r.

(b) Find r (rate) when d (distance) = 170 mi. and t (time) $= 4\frac{1}{4}$ hr.

10. Fig. 239 shows one span of a telephone wire supported by poles at A and B. In the formulas which follow, L is the length (in feet) at the base between the poles, l is the actual length (in feet) of the wire in the span, d is the sag (in feet) from the horizontal position at the center of the wire, w is the weight (in pounds) of a foot of wire, s is the pull (in pounds) with which each wire is stretched.

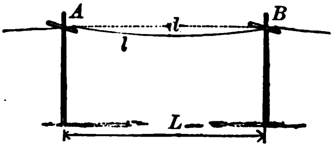


FIG. 239

(a) Given $d = \frac{l^2 w}{8s}$, calculate the sag in a wire 110 ft. long, weighing 0.075 lb. per foot and stretched with a 300-pound pull.

(b) Given $s = \frac{l^2 w}{8d}$, with what pull has a wire weighing 0.075 lb. per foot been stretched if it sags 5 in. and the actual length of the wire between the poles is 100 ft.?

(c) Given $L = l + \frac{8d^2}{3l}$, what is the actual length of a wire which sags 20 in. between poles which are 100 ft. apart?

M.re BLAISE PASCAL DECEDE LE 19e AOUST 1662 AGE DE 39 ANS
Stepha. Desrochers
Sculpsit . 1697 .

Blaise Pascal (1623–1662) was one of the most remarkable scientists that France ever produced. He is said to have discovered by himself, as a child, most of the propositions in the first book of Euclid's famous geometry. At the age of sixteen he wrote a work on conic sections (figures formed by cutting a cone by a plane), and at the age of nineteen he invented a computing machine, — the first of the type which developed into the machines now so widely used in this country. By the time he was twenty-five he was known all over Europe as one of the greatest physicists and mathematicians. He then suddenly decided to devote the rest of his life to the study of philosophy and religion, and in these subjects also he became very noted

CHAPTER V

LINES AND PLANES IN SPACE. DIHEDRAL ANGLES

224. *Postulates.* We shall accept the following assumptions without proof:

1. *A plane is unlimited in extent.*

NOTE. Although we make this assumption, we shall, in the geometrical drawings of this text, represent a plane by some kind of a polygon, usually a parallelogram. The parallelogram is used because it is more pleasing to the eye and represents the situation satisfactorily.

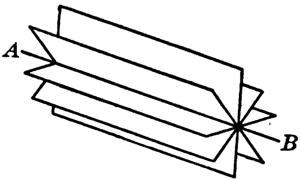


FIG. 240

2. *Through any straight line an unlimited number of planes can be passed* (see Fig. 240).

3. *If a plane is revolved about any straight line in it as an axis, the plane may be made to pass through any point in space.*

4. *If two planes intersect they must have at least two points in common.*

225. Determining a plane. A plane is said to be *determined* by certain points and lines if that plane and no other plane contains the points and lines. The student will remember that a straight line is determined (fixed) by two conditions; namely, by passing it through two given points, or by passing it through a given point in a certain direction.

If one point only of a plane is fixed, the plane is free to take any direction in space. If two points of the plane are fixed, these two points will determine a line about which the plane is free to revolve; that is, this line serves as an axis about which the plane can be made to rotate. But if three points of the plane, which are not collinear, are fixed, then it is evident that the plane is not free to move and hence is determined.

The preceding discussion, which we have made more or less informal, makes it possible for us to establish the truth of the following fundamental theorem:

226. *Theorem.* *A plane is determined by*

1. *Three points which are not collinear.*
2. *A straight line and a point outside the line.*

Proof. The proof of the first part is shown in Art. 225. The truth of the last statement is evident if we remember that any two points on the line may be chosen as two fixed points. Then these two points and the point outside the line determine the plane, as shown above.

NOTE. We may apply the principles of plane geometry to a given system of points and lines in space only when a single plane can be passed through the system of points and lines.

227. *Theorem.* *The intersection of two planes is a straight line.*

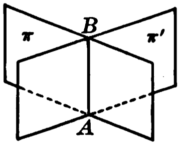


FIG. 241

Given two intersecting planes π and π'.

To prove that *AB, the intersection of the planes π and π', is a straight line.*

Proof

STATEMENTS	REASONS
1. Let A and B be any two points common to the planes π and π', and draw AB.	1. Obvious.
2. Then every point in AB lies both in π and in π'.	2. If a line has two of its points in a plane, it lies wholly in that plane.
3. ∴ AB is common to the planes π and π'.	3. It lies in both.
4. Moreover, no point not on AB can be common to both planes, for the two planes would then coincide.	4. One and only one plane can be passed through three non-collinear points.
5. ∴ the intersection of two planes is a straight line.	

EXERCISES

1. If a sheet of paper is folded, why is the crease straight?

2. In general, in how many points will three planes intersect?

3. What can be said of the intersection of four or more planes in space?

4. Can two pencils be held in a position such that a plane cannot be passed through them? State the general fact about a plane that is illustrated by your answer.

5. In general, can a plane be passed through four or more given points in space? In general, can a plane be passed through three lines all of which pass through a common point in space?

6. If any number of planes pass through a common point P, the line of intersection of each pair of the planes passes through P.

7. A plane is determined by two intersecting lines.

8. Can there be two straight lines that are not parallel and that do not meet? Illustrate your answer.

9. The locus of a point in space equidistant from the ends of a line segment is the plane perpendicular to the line segment at its midpoint.

228. Skew lines. In general, two straight lines in space are not coplanar and do not intersect; such lines are said to be *skew* to one another.

229. Skew polygon. A figure bounded by a closed broken line consisting of any number of lines not all of which are coplanar is called a *skew polygon.* Thus, $ABCD$ (Fig. 242) is a *skew quadrilateral.*

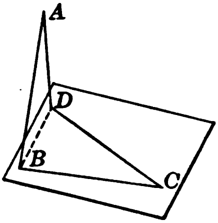


FIG. 242

EXERCISES

1. Show that two parallel lines determine a plane.

2. Name the conditions necessary for the determination of a plane.

3. Why is a surveyor's transit or a photographer's camera supported by three legs instead of four?

4. How do builders of concrete sidewalks get a plane surface? Explain your answer.

5. When a carpenter changes the rough surface of a board to a plane (smooth) surface, what geometric principle is involved?

6. Find four points in the classroom through which a plane may be passed. Find four points through which a plane cannot be passed.

7. How many planes are determined by four points which are not coplanar?

8. Show that a plane cannot be passed through two skew lines.

9. Prove that a straight line can intersect a plane in but one point.

10. Show that a triangle is always a plane figure. Show that a quadrilateral is always a plane figure if two of its sides are parallel.

11. Draw a skew quadrilateral. How many planes are determined by its sides?

12. Draw the diagonals of the skew quadrilateral in Ex. 11.

13. Show how to construct through a given point a straight line which shall intersect each of two given skew lines.

14. Show that any number of lines can be drawn so as to intersect each of three given skew lines.

15. Show that through a given straight line any number of planes can be passed.

SUGGESTION. Let l and l_1 be two skew lines. The line l and each point on l_1 determine a plane.

230. Direct method of proof. In most of the work thus far undertaken the student can see that in proving the propositions the method of reasoning consisted of putting together known things, such as axioms, definitions, propositions previously proved, the hypothesis, and so on, in order that new truths might be derived step by step until the conclusion was reached, or until the proposition was proved. This method is called the *direct method of proof.*

231. Indirect method of proof. Another method, which is often used when the direct method is hard to apply or does not seem applicable, is the so-called *indirect method of proof.* We shall now illustrate this method of proof by proving a theorem.

232. *Theorem.* ***If each of two lines is parallel to a third line, the lines are parallel to each other.***

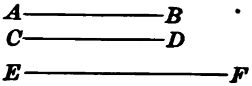


FIG. 243

Given AB parallel to EF, and CD parallel to EF.

To prove that $AB \parallel CD$.

Proof

STATEMENTS	REASONS
1. Assume that AB is not $\parallel$ CD.	1. AB must be either parallel to CD or not parallel to CD, and we try the latter.
2. Then AB and CD, if extended, must meet at some point P.	2. Two nonparallel lines have one and only one common point.
3. Then PA would be $\parallel$ FE, and PC would be $\parallel$ FE.	3. By hypothesis.
4. This would give us two lines passing through P $\parallel$ FE, which is impossible.	4. Through a point outside a given line only one line can be drawn parallel to the given line.
5. The assumption that AB is not $\parallel$ CD is incorrect, and $\therefore AB \parallel CD$.	5. It is the only other alternative.

233. Reductio ad absurdum. The method used in proving the preceding theorem consists in supposing that what is to be proved is not true, and then proving that the supposition leads to a clear contradiction of a known truth; that is, to an absurdity. Writers on mathematics have designated this method by the Latin term *reductio ad absurdum* (meaning "reduction to an absurdity"). It is a powerful method of proof in many instances in ordinary life as well as in mathematics.

We may sum up the indirect method of proof in the following steps:

1. Assume that what you want to prove is not true.
2. Show by reasoning that the assumption leads to a contradiction of known truths or to an absurdity.
3. This proves that the assumption is false.
4. Therefore the conclusion to be reached is correct, and the theorem is true.

EXERCISES

Prove the following theorems, where possible, by the indirect method:

1. At a given point on a given line only one perpendicular can be drawn to the line.

2. Through a given point not on a given line one and only one perpendicular to the line can be drawn.

3. Two straight lines perpendicular respectively to two intersecting straight lines must intersect.

4. Lines drawn from the vertices of two angles of a triangle and lying within the sides of these angles must intersect.

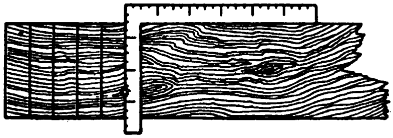


FIG. 244

5. The carpenter uses the principle of Ex. 3 when he uses his steel square to mark parallel lines on a piece of lumber. Explain how the square is used for this purpose (see Fig. 244).

234. *Theorem.* *Through a given point P in space outside a given line one and only one line can be drawn parallel to the given line.*

SUGGESTION. Let P be the given point and AB the given line. Show first that a plane π can be passed through P and AB and that any line through $P \parallel AB$ must lie in π. Then, obviously, one and only one line can be drawn in π through $P \parallel AB$.

235. Theorem. *If a plane intersects one of two parallel lines, it intersects the other line also.*

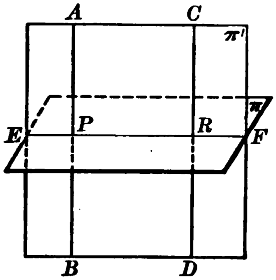


FIG. 245

Given AP parallel to CR, and the plane π intersecting AP in P.

To prove that *π intersects CR.*

Proof

STATEMENTS	REASONS
1. AP and CR determine a plane π'.	1. Two parallel lines determine a plane.
2. Since π and π' have the point P in common, they intersect in a line EF passing through P.	2. The intersection of two planes is a straight line.
3. Also, since EF lies in the same plane with AP and CR and intersects AP at P, it must also intersect CR in some point, as R.	3. If a line is coplanar with two parallel lines and intersects one of the lines, it must intersect the other line also.
4. Since π contains all points of EF, it contains the point P.	4. The point P lies on EF.
5. The plane π, however, cannot contain CR without also containing AP.	5. Obvious.
6. $\therefore$ since π contains the point R on CR but does not contain CR, it must intersect CR.	6. The line and the plane have a common point R; but the plane does not, for example, contain the point C on CR.

236. Corollary. *If a plane contains only one of two parallel lines, it can never intersect the other line.*

237. Lines parallel to a plane; parallel planes. A straight line is said to be parallel to a plane if it never meets the plane however far it is produced. The plane is also said to be parallel to the line.

Two planes are parallel if they never meet however far they are produced.

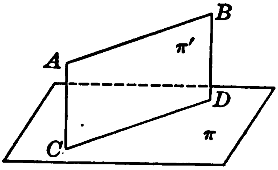


FIG. 246

238. Corollary 1. *If two lines are parallel, every plane containing one and only one of the lines is parallel to the other line.*

SUGGESTION. Suppose AB is not $\parallel$ CD, and then show that if that is true they would meet at some point common to planes π and π'. Therefore the point would be on CD, which can be shown to be impossible.

239. Corollary 2. *If a given line and a given plane are parallel, and if a second line is drawn parallel to the given line through any point of the plane, the second line will lie in the plane.*

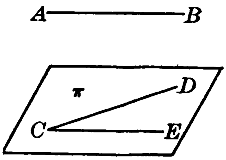


FIG. 247

240. Corollary 3. *Through either of two skew lines one and only one plane can be passed parallel to the other line.*

SUGGESTION. Let AB and CD (Fig. 247) be the given lines. Pass plane π through CD and any line CE which is $\parallel$ AB. Then show that $\pi \parallel AB$ and that it is the only plane that is $\parallel AB$.

EXERCISE

Show that (a) a line and a plane may have no point in common, (b) may have one point in common, (c) and that the line may lie wholly in the plane.

241. Theorem. *A line parallel to a plane is parallel to the intersection of this plane with any plane passing through the line.*

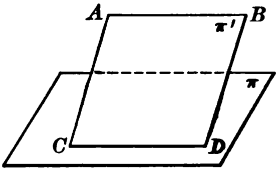


FIG. 248

Given the line AB parallel to the plane π.

To prove that *any plane π' containing AB intersects π in a line CD that is parallel to AB.*

Proof

STATEMENTS	REASONS
1. AB and CD cannot meet.	1. AB is parallel to π, and if it met CD, it would meet π also.
2. AB and CD are coplanar.	2. Any two lines in the same plane are coplanar.
3. $\therefore AB \parallel CD$.	3. By definition; that is, they are coplanar and never meet.

242. Corollary 1. *If two intersecting lines are each parallel to a given plane, the plane determined by them is parallel to the given plane.*

SUGGESTION. Let AB and AC each be $\parallel$ the plane π (Fig. 249). Then show that if the plane of AB and AC intersects π an absurdity results.

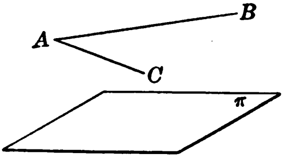


FIG. 249

243. Corollary 2. *If two planes are parallel, every line in one plane is parallel to the other plane.*

244. Corollary 3. *If a line intersects one of two parallel planes, it intersects the other plane.*

245. *Theorem.* *Through any point outside a given plane one and only one plane can be passed parallel to the given plane.*

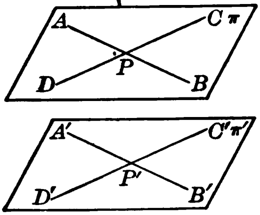


FIG. 250

Given the plane π' and a point P outside the plane π'.

To prove that *one and only one plane π can be passed through P parallel to the plane π'.*

Proof

STATEMENTS	REASONS
1. Through P', any point of π', two lines $A'B'$ and $C'D'$ can be drawn.	1. Obvious.
2. Through P a line AB can be drawn $\parallel A'B'$, and a line CD can be drawn $\parallel C'D'$.	2. Through a point outside a given line a line may be drawn parallel to the given line.
3. $AB \parallel \pi'$, and $CD \parallel \pi'$.	3. If a plane contains only one of two parallel lines, it cannot meet the other line.
4. AB and CD determine a plane π which is $\parallel \pi'$.	4. If each of two intersecting lines is parallel to a plane, the plane determined by them cannot meet the given plane.
5. Now if another plane could be passed through $P \parallel \pi'$, it would be $\parallel A'B'$ and $C'D'$.	5. If two planes are parallel, every line in one plane is parallel to the other plane.
6. $\therefore$ it would contain AB and CD.	6. By Art. 239.
7. But this is impossible.	7. Only one plane is determined by two intersecting lines.
8. $\therefore$ the theorem is true.	

246. Corollary. *If a plane intersects one of two parallel planes, it intersects the other plane.*

EXERCISES

1. What is the locus of a line which is drawn through a given point parallel to a given plane?

2. Find the locus of a point in a given plane, the point being equidistant from two given points not in the plane.

3. What is the locus of a point equidistant from three given points which are not collinear? from four given points, no three of which are collinear?

4. Find the locus of a point which is equidistant from two given parallel planes and which is at the same time equidistant from two given points.

5. Two planes parallel to a third plane are parallel to each other.

6. A line parallel to each of two intersecting planes is parallel to their intersection.

247. Perpendicular to a plane. If a straight line is drawn to a plane and is perpendicular to every straight line passing through its foot and lying in the plane, the line is said to be perpendicular to the plane. If a line is perpendicular to a plane, the plane is also said to be perpendicular to the line. Thus, in Fig. 251, if PR is perpendicular to π, it is perpendicular to all the lines AB, CD, EF, HG, and so on.

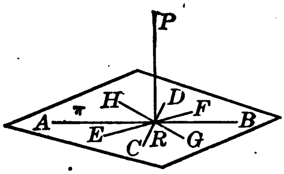


FIG. 251

EXERCISES

1. Pick out three lines in the classroom that are perpendicular to some plane.

2. Show why the edge of a rectangular parallelepiped is perpendicular to one of the faces.

248. Theorem. *If a line is perpendicular to each of two lines at their point of intersection, it is perpendicular to the plane determined by the two lines.*

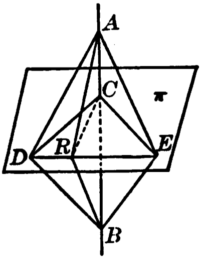


FIG. 252

Given the line *AC* perpendicular to *DC* and *CE* at *C*.

To prove that $AB \perp \pi$.

Proof

STATEMENTS	REASONS
1. Let CR be any line in plane π passing through C, and draw DE cutting CD, CR, and CE.	1. Obvious.
2. Lay off $CB = AC$ and connect A and B with D, R, and E.	2. A line may be laid off equal to another, and only one straight line may be drawn between two points.
3. Then DC and EC are each $\perp$ AB at its midpoint.	3. It was given that DC is perpendicular to AB and that CE is perpendicular to AB, and AC and CB are equal by construction.
4. $\therefore AD = DB$, and $AE = EB$.	4. The perpendicular bisector of a line segment is the locus of all points equidistant from the ends of the line segment.
5. Then, since DE is common to the $\triangle$ ADE and DBE, the triangles are congruent.	5. If three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent.

Proof (*Continued*)

STATEMENTS	REASONS
6. $\therefore \angle EDA = \angle EDB$.	6. Homologous parts of congruent triangles are equal.
7. Then, since DR is common to the $\triangle$ ADR and DBR, the triangles are congruent.	7. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
8. $\therefore AR = RB$.	8. Homologous parts of congruent triangles are equal.
9. $\therefore AB \perp RC$.	9. If two points of one line are equidistant from two points of another line, the first line is the perpendicular bisector of the second line.
10. Since $AB \perp RC$ and RC is any line in the plane π passing through C, AB must be $\perp$ the plane π.	10. A line is perpendicular to a plane if it is perpendicular to any line drawn through its foot in the plane.

249. *Construction problem.* *At a given point in a given plane to construct a perpendicular to the plane.*

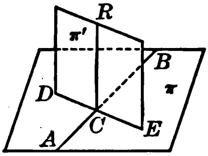


FIG. 253

SUGGESTION. Draw AB, any line in the plane π, through the given point C. Construct plane $\pi' \perp AB$ at C and cutting π in the line DE. Then in plane π' draw $RC \perp DE$ at C. Then show that $RC \perp DE$ and AB at their point of intersection and that it is $\perp$ plane π.

250. *Construction problem.* *Through a given point outside a given plane to construct a line perpendicular to the given plane.*

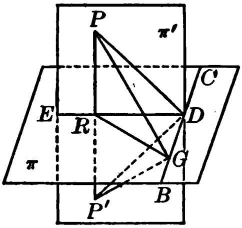


FIG. 254

SUGGESTION. In plane π draw a line, as BC, and let π' be a plane through the given point $P \perp BC$ and cutting π in ED and BC in D. Draw $PR \perp ED$, and in plane π draw any line RG from R to BC. Produce PR its own length to P', and draw PD, PG, $P'D$, and $P'G$. Then, since $GD \perp \pi'$, the $\measuredangle PDG$ and $P'DG$ are rt. $\measuredangle$, DG is common, and $PD = P'D$; hence $\triangle PDG \cong \triangle P'DG$. Then $PG = P'G$, and RG is the perpendicular bisector of PP'. Then show that $PR \perp \pi$.

251. *Corollary 1.* *A perpendicular is the shortest line (distance) from a point to a plane.*

252. *Corollary 2.* *The locus of points in a plane at a given distance from a given point outside the plane is a circle whose center is the foot of the perpendicular from the point to the plane.*

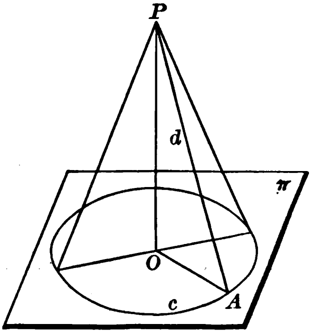


FIG. 255

EXERCISE

Can more than one perpendicular be drawn to a line at a point of the line in a plane? in space?

253. Theorem. *At a given point in a given plane only one perpendicular line can be erected.*

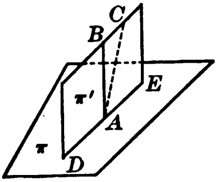


FIG. 256

SUGGESTION. Let AB be ⊥ plane π at A, and suppose that another perpendicular AC could be erected to the plane π at A. Then imagine a plane π' passed through BA and AC. This plane would intersect the plane π in a line DE. Show that both BA and AC would then be ⊥ DE, which is impossible.

254. Theorem. *From a given point outside a given plane only one line can be drawn perpendicular to the plane.*

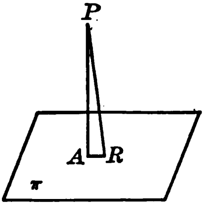


FIG. 257

SUGGESTION. If two perpendiculars, as PR and PA, could be drawn from P to the plane π, then the $\triangle RPA$ would contain two right angles. Show that this is impossible.

EXERCISES

1. What is the locus of all points equidistant from a circle?

2. Could a tent pole be sufficiently braced with three guy ropes? with two? Give reasons for your answers.

255. Theorem. *All the perpendiculars that can be drawn to a given line at a given point must lie in a plane which is perpendicular to the given line at the given point.*

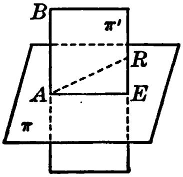


FIG. 258

Given the plane π perpendicular to *AB* at *A*.

To prove that *any line ⊥ AB at A lies in the plane π.*

Proof

STATEMENTS	REASONS
1. Let AE be any line ⊥ AB at A and let the plane π′, determined by AB and AE, cut the plane π in the line AR.	1. Two lines determine a plane, and two planes intersect in a straight line.
2. Then $AB \perp AR$.	2. If a line is perpendicular to a plane, it is perpendicular to every line drawn through its foot in the plane.
3. In the plane π′ only one perpendicular can be drawn to AB at A.	3. Only one perpendicular can be drawn to a given line at a given point.
4. ∴ AR and AE must coincide, and AE lies in plane π.	4. Obvious.
5. Since AE represents any line, the theorem is true.	

256. *Corollary 1.* *At a given point in a given line only one plane can be drawn perpendicular to the given line.*

SUGGESTION. Show that this follows at once from the theorem of Art. 255.

257. Corollary 2. *Through a given point outside a given line only one plane can be drawn perpendicular to the given line.*

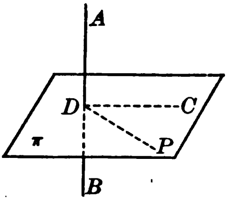


FIG. 259

SUGGESTION. Take AD (Fig. 259) as the given line and P as the given point. Draw $PD \perp AD$, and $CD \perp AD$. Then show that the plane π is determined by these lines and that it is the only perpendicular plane that can be drawn.

EXERCISES

1. In setting a piece of studding perpendicular to the floor, how does a carpenter place his steel square to determine the position of the piece?

2. If the spokes of a wagon wheel are perpendicular to the hub, why is it that the wheel in turning always traces a plane surface?

258. Theorem. *Two lines perpendicular to the same plane are parallel.*

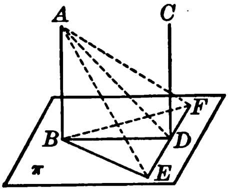


FIG. 260

Given AB and CD, each perpendicular to the plane π.

To prove that $AB \parallel CD$.

Proof

STATEMENTS	REASONS
1. In the plane π draw BD; then draw $EF \perp BD$, making $DE = DF$.	1. A perpendicular can be drawn to a line at a given point, and a line segment can be made equal to another.
2. Then $BE = BF$.	2. Any point on the perpendicular bisector of a line segment is equidistant from the ends.

Proof (*Continued*)

STATEMENTS	REASONS
3. $\therefore AE = AF$.	3. Oblique lines drawn from a point on a perpendicular to a plane and meeting the plane at points equidistant from the foot of the perpendicular are equal.
4. Then $AD \perp EF$.	4. A line drawn from the vertex of an isosceles triangle to the midpoint of the base is perpendicular to the base.
5. We now have $EF \perp DA$, DB, and DC.	5. Obvious.
6. $\therefore DA$, DB, and DC are coplanar.	6. All the perpendiculars that can be drawn to a given line at a given point lie in the plane perpendicular to the given line at the given point.
7. $\therefore AB$ and CD are coplanar.	7. If two points of a line lie in a plane, the line lies wholly in the plane.
8. Since AB and CD are each $\perp BD$, we know that $AB \parallel CD$.	8. If two coplanar lines are perpendicular to the same line, they are parallel to each other.

259. Corollary 1. *If one of two parallel lines is perpendicular to a plane, the other line is perpendicular to the plane.*

SUGGESTION. Let AB (Fig. 261) be $\parallel CD$ and $\perp$ the plane π. Then if CD is not $\perp$ the plane π, $C'D$ is $\perp$. Show that this is impossible.

260. Corollary 2. *In space, if two lines are parallel to a third line they are parallel to each other.*

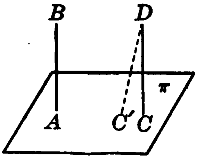


FIG. 261

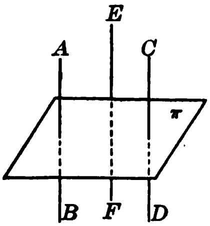


FIG. 262

SUGGESTION. Let π (Fig. 262) be $\perp AB$. Prove $\pi \perp CD$ and EF.

EXERCISES

1. Equal oblique lines drawn to a plane from a given point outside the plane make equal angles with the lines drawn in the plane from the points where the oblique lines intersect the plane to the foot of the perpendicular from the given point to the plane.

2. If through the foot of a perpendicular to a plane a line is drawn perpendicular to a given line in the plane, the line from the intersection of the two lines to any point on the perpendicular is perpendicular to the given line in the plane.

SUGGESTION. Take $BC = BD$ (Fig. 263). Draw AD, AC, PD, and PC. Compare AC and AD, and then compare PD and PC.

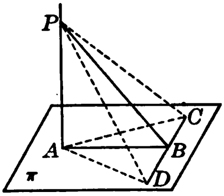


FIG. 263

3. Through a given line which is parallel to a given plane a number of planes are passed intersecting the given plane. Prove that the lines of intersection with the given plane are all parallel to each other.

4. If two points are equidistant from a plane and on the same side of it, the line joining them is parallel to the plane.

5. If one of two parallel lines is parallel to a plane, the other is parallel to the plane unless it lies in the plane.

6. If the midpoints of the sides of a quadrilateral in space are joined in order, the lines joining the midpoints form a parallelogram.

7. If a line is perpendicular to one of two parallel planes, it is perpendicular to the other plane.

8. If a line is perpendicular to a line of a plane, is it perpendicular to the plane?

9. From a point P, 3 in. from a plane π, a line PB is drawn so that B lies in the plane π. If $PB = 5$ in., what is the distance of B from the foot of the perpendicular D drawn from P to a point D in the plane π?

261. Theorem. *If one of two parallel planes is perpendicular to a line, the other plane is perpendicular to the line.*

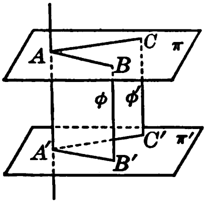


FIG. 264

SUGGESTION. Let π be $\perp$ AA' and $\parallel$ π', and pass the planes ϕ and ϕ' through AA', cutting the planes π and π' in AB, AC, $A'B'$, and $A'C'$ respectively. Then show that $AB \parallel A'B'$ and $AC \parallel A'C'$. But $AA' \perp AB$ and AC. Then show that AA' must be $\perp$ $A'B'$ and $A'C'$ and hence $\perp$ π'.

262. Corollary 1. *Through a given point one and only one plane can be passed parallel to a given plane.*

SUGGESTION. Show how a plane through the point A (Fig. 264) and $\perp$ AA' is related to the plane π', and then show that it is the only plane that can be passed through $A \perp AA'$.

263. Corollary 2. *The locus of a point equidistant from two parallel planes is a plane perpendicular to a line which is perpendicular to the planes and which is the bisector of the line segment cut off by the planes.*

264. Corollary 3. *The locus of a point equidistant from two parallel lines is a plane perpendicular to a line which is perpendicular to the given lines and which bisects the line segment cut off by them.*

EXERCISES

1. Are two lines, each of which is parallel to a given plane, necessarily parallel to each other? Can they be parallel to each other? Illustrate your answers.

2. Can two nonintersecting lines lie in a plane parallel to another plane?

265. Theorem. *If two angles which are not coplanar have their sides parallel right to right and left to left and extending in the same direction, the angles are equal and their planes are parallel.*

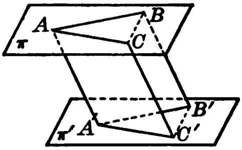


FIG. 265

Given the angles A and A', with AB parallel to $A'B'$, AC parallel to $A'C'$, and extending in the same direction.

To prove that $\angle A = \angle A'$, *and* $\pi \parallel \pi'$.

Proof

STATEMENTS	REASONS
1. Draw AA' and lay off $AB = A'B'$ and $AC = A'C'$. Then draw BC, $B'C'$, BB', and CC'.	1. A straight line can be drawn between two points, and a line segment can be made equal in length to a given line.
2. Since $AB = A'B'$,	2. By construction.
3. and $AB \parallel A'B'$,	3. Given.
4. $AA'B'B$ is a parallelogram.	4. If a pair of opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.
5. $\therefore AA' \parallel BB'$, and $AA' = BB'$.	5. The opposite sides of a parallelogram are equal and parallel.
6. In like manner, $AA' \parallel CC$, and $AA' = CC'$.	6. For the same reason.

Proof (*Continued*)

STATEMENTS	REASONS
7. $\therefore CC' \parallel BB'$, and $CC' = BB'$.	7. If two lines are equal and parallel to the same line, they are equal and parallel to each other.
8. $\therefore \triangle ABC \cong \triangle A'B'C'$.	8. If three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent.
9. $\therefore \angle A = \angle A'$.	9. Homologous parts of congruent triangles are equal.
10. $\pi \parallel A'C'$, and $\pi \parallel A'B'$.	10. If two lines are parallel, a plane containing one and only one of the lines is parallel to the other line.
11. $\therefore \pi \parallel \pi'$.	11. If two intersecting lines are parallel to a given plane, their plane is parallel to the given plane.

EXERCISE

What is the theorem of plane geometry which is analogous to the preceding theorem? Why does not the proof for it apply here?

266. Theorem. *Two planes perpendicular to the same line are parallel to each other.*

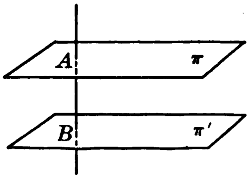


FIG. 266

SUGGESTION. Use the indirect method with Fig. 266.

267. Theorem. *If two parallel planes are cut by a third plane, the lines of intersection are parallel.*

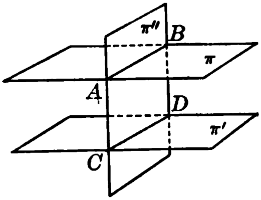


FIG. 267

SUGGESTION. Show by the indirect method that AB and CD (Fig. 267) can never meet.

268. Corollary 1. *Parallel line segments included between parallel planes are equal.*

SUGGESTION. Show that $ABCD$ (Fig. 268) is a parallelogram and from this show that $AD = BC$.

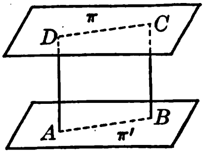


FIG. 268

269. Corollary 2. *Two parallel planes are everywhere equidistant.*

SUGGESTION. Drop perpendiculars from any points in one plane to the other plane, and prove that these perpendiculars are parallel and equal.

EXERCISES

1. Find classroom illustrations of the two preceding theorems.

2. If a given line is parallel to a given plane, the intersection of the plane with any plane through the given line is parallel to the given line.

3. If a given line is parallel to a given plane, a line drawn parallel to the given line through any point of the plane lies in the plane.

4. What is the locus of a point which is equidistant from two points M and N and which is also equidistant from two other points P and Q?

270. Dihedral angles. If a plane π (Fig. 269) revolves about a line AB in the plane, the amount of rotation made by the plane in reaching the position π' is called a *dihedral angle.*

The planes π and π' are called the *faces* of the dihedral angle, and the line AB is called the *edge* of the dihedral angle.

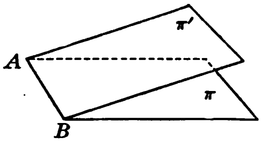


FIG. 269

271. How to read a dihedral angle. A dihedral angle is usually read by naming the letters which designate its edge, or those of an edge and an additional point in each face. Thus, in Fig. 270 we should designate the dihedral angle as AB or D-BA-C. Sometimes a single letter is used to denote the dihedral angle, as d (Fig. 270).

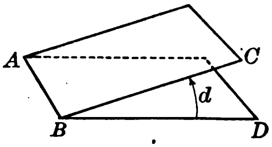


FIG. 270

272. Size of a dihedral angle. The size of a dihedral angle depends only upon the amount of turning made by the rotating plane and not upon the extent of the faces. The student should recognize the analogy in this respect to the ordinary plane angle.

273. Plane angle of a dihedral angle. The plane angle formed by two straight lines, one in each plane and perpendicular to the edge of a dihedral angle at the same point, is called the *plane angle* of the dihedral angle. For example, $\angle ABC$ (Fig. 271) is the plane angle of the dihedral $\angle MN$ if the lines AB and CB are each perpendicular to the edge MN.

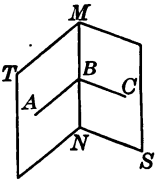


FIG. 271

274. Kinds of dihedral angles. A dihedral angle is said to be *acute*, *right*, *obtuse*, *reflex*, *vertical*, *complementary*, or *supplementary*, according as its plane angle is acute, right, obtuse, reflex, vertical, complementary, or supplementary.

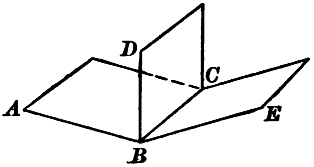


FIG. 272

Adjacent dihedral angles are those dihedral angles which have a common edge and a common face between them. Again, the student should notice the analogy to adjacent plane angles. Thus, in Fig. 272 *A-BC-D* and *D-BC-E* are adjacent dihedral angles.

275. Perpendicular planes. If two planes intersect so as to form a right dihedral angle, each of the planes is said to be perpendicular to the other.

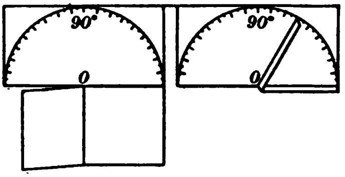


FIG. 273

276. Measuring a dihedral angle. An instrument for measuring the number of degrees in dihedral angles can be made by cutting and folding a piece of heavy cardboard, as shown in Fig. 273.

277. ***Theorem.*** *All plane angles of a dihedral angle are equal.*

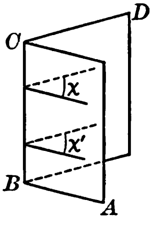


FIG. 274

SUGGESTION. Show that the sides of $\angle s\ x$ and x' (Fig. 274) are parallel left to left and right to right and extend in the same direction. Then show that they are equal.

278. Theorem. *Two dihedral angles are equal if their plane angles are equal.*

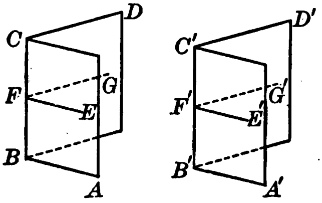


FIG. 275

Given the dihedral angles BC and $B'C'$ and their plane angles EFG and $E'F'G'$.

To prove that $BC = B'C'$.

Proof

STATEMENTS	REASONS
1. Imagine dihedral $\angle BC$ placed on dihedral $\angle B'C'$ so that $\angle EFG$ will coincide with its equal $\angle E'F'G'$; then $C'F' \perp F'G'$ and $F'E'$.	1. By definition of a plane angle of a dihedral angle.
2. $\therefore C'F' \perp$ plane $E'F'G'$.	2. If a line is perpendicular to two lines at their point of intersection, it is perpendicular to the plane of these lines.
3. $\therefore CF$ must coincide with $C'F'$.	3. Only one perpendicular can be erected at a given point in a plane.
4. Then the faces BCD and $B'C'D'$ coincide.	4. They have in common the two intersecting lines $B'C'$ and $F'G'$.
5. In like manner, the faces CBA and $C'B'A'$ coincide.	5. For the same reason.
6. $\therefore$ the dihedral $\angle$s BC and $B'C'$ are equal.	6. They coincide throughout.

279. Corollary 1. *All right dihedral angles are equal.*

280. Corollary 2. *Vertical dihedral angles are equal.*

EXERCISES

1. State and prove the converse of the preceding theorem.

2. Are there such angles as equal straight dihedral angles? Give the reason for your answer.

3. Dihedral angles which are complements or supplements of the same or of equal dihedral angles are equal.

4. Prove that if two parallel planes are cut by a transverse plane,

(a) The corresponding dihedral angles are equal.
(b) The alternate-interior dihedral angles are equal.
(c) The interior dihedral angles on the same side of the transverse plane are supplementary.

5. State and prove the converse of Ex. 4.

6. Prove that the planes which bisect each of a pair of vertical dihedral angles are perpendicular.

281. Theorem. *If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.*

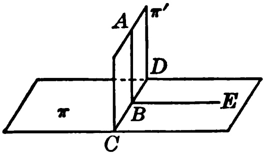


FIG. 276

Given the plane π perpendicular to π', and AB perpendicular to CD.

To prove that $AB \perp \pi$.

Proof

STATEMENTS	REASONS
1. In plane π draw $BE \perp CD$.	1. At a point on a given line one and only one perpendicular can be drawn to the line.
2. Then $\angle EBA$ is the plane angle of the dihedral $\angle CD$ and is therefore a right angle.	2. The plane angle of a dihedral angle may be taken as the measure of the dihedral angle.
3. $\therefore AB \perp BE$.	3. By definition of a perpendicular.
4. But $AB \perp CD$.	4. Given.
5. $\therefore AB \perp \pi$.	5. If a line is perpendicular to two lines at their point of intersection, it is perpendicular to the plane determined by them.

282. ***Corollary 1.*** *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other plane.*

SUGGESTION. Show that the line AB, which was drawn $\perp CD$ at B in the plane π' (Fig. 276), is $\perp \pi$ and that only one such perpendicular can be erected at B to the plane π.

283. ***Corollary 2.*** *If two planes are perpendicular to each other, a perpendicular to the first plane from any point in the second will lie in the second plane.*

SUGGESTION. Show that a line AB drawn $\perp CD$ in the plane π' is $\perp \pi$ and that it is the only perpendicular.

284. ***Corollary 3.*** *A plane perpendicular to one of two parallel planes is perpendicular to the other plane.*

EXERCISES

1. The sum of two adjacent dihedral angles formed by two intersecting planes is 180°.

2. If the sum of two adjacent dihedral angles is 180° the exterior faces lie in the same plane.

285. Theorem. *If a line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.*

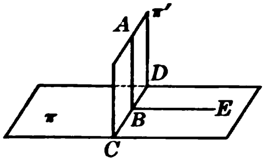


FIG. 277

Given AB perpendicular to the plane π at B, and the plane π' through AB cutting π in CD.

To prove that $\pi' \perp \pi$.

Proof

STATEMENTS	REASONS
1. Draw $BE \perp CD$ in plane π.	1. At a point on a given line one and only one perpendicular can be drawn to the line.
2. Since $AB \perp \pi$,	2. Given.
3. $AB \perp CD$.	3. By definition of a perpendicular to a plane.
4. $\therefore \angle ABE$ is the measure of the dihedral $\angle CD$.	4. The plane angle of a dihedral angle may be taken as the measure of the dihedral angle.
5. But $\angle ABE$ is a right angle.	5. By definition of a perpendicular to a plane.
6. $\therefore \pi' \perp \pi$.	6. By definition of perpendicular planes.

EXERCISES

1. Show that through a line perpendicular to a given plane any number of planes may be drawn perpendicular to the given plane.

2. A plane which is perpendicular to the edge of a dihedral angle is perpendicular to each face of the dihedral angle.

3. Through a point within a dihedral angle a plane may be passed perpendicular to each face.

4. If two lines are perpendicular to each other, is any plane through one of the lines perpendicular to the other? Prove your answer.

5. Suppose that three lines are perpendicular to each other at a point common to each line. What relation do the three planes determined by the three given lines bear to each other? Prove your answer.

286. ***Theorem.*** *If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.*

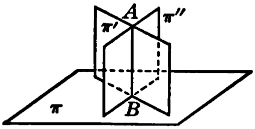


FIG. 278

Given the planes π' and π'' intersecting in the line AB and each perpendicular to the plane π.

To prove that $AB \perp \pi$.

Proof

STATEMENTS	REASONS
1. At B, the point common to π, π', and π'', draw a line $\perp$ π.	1. At a point in a plane one and only one perpendicular to the plane can be drawn.
2. This perpendicular must lie in π''.	2. If through a point in one of two perpendicular planes a line is drawn perpendicular to the other, the line lies in the first plane.
3. This perpendicular must also lie in π'.	3. For the same reason.
4. $\therefore$ this perpendicular must be the intersection of π' and π''.	4. If a line lies in two planes, it coincides with the line of intersection of the planes.
5. $\therefore$ this perpendicular coincides with AB.	5. AB is given as the intersection of π' and π''.

287. *Theorem.* *Through a given line not perpendicular to a given plane one and only one plane can be passed perpendicular to the plane.*

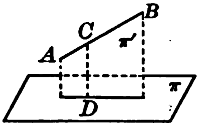


FIG. 279

Given the line *AB* not perpendicular to the plane π.

To prove that *through AB one and only one plane can be drawn ⊥ π.*

Proof

STATEMENTS	REASONS
1. From any point C on AB draw $CD \perp \pi$.	1. From a point outside a plane one and only one perpendicular can be drawn.
2. Then the plane π' determined by AB and CD is the required plane.	2. If a line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.
3. π' is the only plane that can be passed through $AB \perp \pi$.	3. If any other plane could be passed through AB perpendicular to π, then π would be perpendicular to AB, the line of intersection, by Art. 286. But this contradicts the hypothesis.
4. ∴ the theorem is true.	

288. Projection of a point on a plane. The foot of the perpendicular from a given point outside a plane to the plane is called the *projection of the point on the plane.*

289. Projection of a line on a plane. The locus of the projections of the points of a given line segment AB on a plane π is called the *projection of the line on the plane.* Thus, in Fig. 280 $A'B'$ is the projection of AB on the plane π.

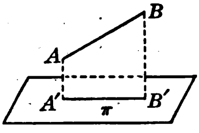


FIG. 280

290. Theorem. *The projection of a straight line not perpendicular to a plane upon the plane is a straight line.*

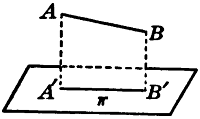


FIG. 281

Given AB, not perpendicular to plane π, and $A'B'$, the projection of AB upon π.

To prove that *$A'B'$ is a straight line.*

Proof

STATEMENTS	REASONS
1. AA' and BB' are each $\perp$ π.	1. By definition of the projection of a point on a plane.
2. $\therefore AA' \parallel BB'$.	2. Two lines perpendicular to the same plane are parallel.
3. $\therefore AA'$ and BB' determine a plane π', which contains AB and cuts plane π in $A'B'$.	3. Two parallel lines determine a plane, and the intersection of the two planes is $A'B'$.
4. $\pi' \perp \pi$.	4. If a line is perpendicular to a plane, any plane passed through that line is perpendicular to the plane.
5. $\therefore$ the perpendiculars to the plane π from all points of AB lie in the plane π'.	5. If two planes are perpendicular to each other, all perpendiculars from a point in the first plane to the second plane lie in the first plane.
6. $\therefore$ the projections of all points of AB are in $A'B'$; that is, $A'B'$ is the projection of AB.	

291. Corollary. *The projection of a straight line perpendicular to a plane upon the plane is a point.*

292. Inclination of a line to a plane. If a straight line cuts a plane, the projection upon the plane of the point of intersection is the point of intersection itself. Thus, when a line meets a plane, it meets it at a point of its projection on the plane. The angle which the line makes with its projection is called the *inclination of the line* to the plane. Ordinarily, therefore, a line makes an acute angle with its projection upon the plane.

293. ***Theorem.*** *The acute angle which a line makes with its projection upon a plane is the least angle which it makes with any line of the plane.*

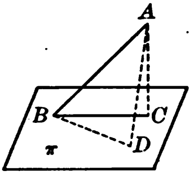


FIG. 282

SUGGESTION. Let BC (Fig. 282) be the projection of AB upon the plane π, and BD any line other than BC in the plane π such that $BD = BC$. Then prove that $\angle ABD > \angle ABC$.

EXERCISES

1. What may be said if AB (Fig. 282) is parallel to the plane π? if AB is perpendicular to the plane π?

2. The projections of two parallel lines upon a plane are parallel.

3. If two parallel lines intersect a plane, their inclinations with the plane are equal.

4. If a straight line cuts two parallel planes, it makes equal angles with the planes.

5. Prove that a straight line and its projection upon a given plane determine a plane which is perpendicular to the given plane.

6. Prove that the ratio of two parallel line segments is equal to the ratio of the projections of these lines upon a plane.

7. Can a curve or a broken line have a straight-line projection upon a plane? Explain your answer.

8. If the projection of a given geometric figure upon a plane is a straight line, then all parts of the given figure are coplanar.

CHAPTER VI

STRAIGHT LINES AND CIRCLES

294. Chord. A *chord* of a circle is a line segment terminated at each end by the circle. Thus, a chord may be thought of as connecting the two ends of an arc.

295. Major arc; minor arc. If an arc and its chord inclose the center of the circle, the arc is called a *major arc*; otherwise, it is called a *minor arc.* Thus it is seen that the chord of a circle is the line that separates the major arc from the minor arc.

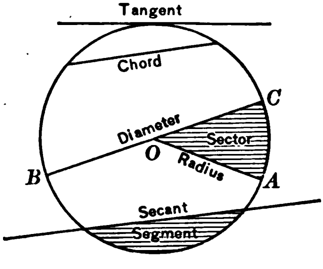


FIG. 283

296. Secant; tangent. A line which cuts a circle in two points is called a *secant.* If a line touches a circle in only one point, no matter how long the line is, it is called a *tangent,* and the point where the line touches the circle is called the *tangent point,* or *point of contact.*

297. Sector. That portion of the area of a circle which is included between two radii and the arc which they intercept is called a *sector.*

298. Segments. The parts into which the area inside a circle is divided by a chord or secant are called *circular segments,* or briefly *segments.*

EXERCISES

1. Explain the difference between a chord and a secant.

2. Show by superposition that two circles which have equal radii are congruent, and conversely.

3. Show that if two circles have equal diameters they are congruent, and conversely.

299. Theorem. *In the same circle or in congruent circles equal arcs have equal chords.*

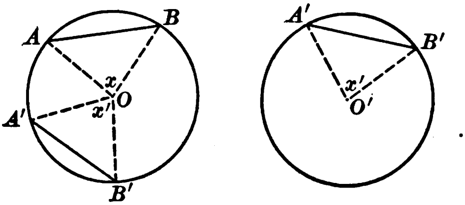


FIG. 284

Given the congruent circles *O* and *O'*, with arc *AB* equal to arc *A'B'*.

To prove that $AB = A'B'$.

Proof

STATEMENTS	REASONS
1. Imagine that ⊙ O' is picked up and placed on ⊙ O so that O' falls on O, and that ⊙ O' is then rotated about O until $A'B'$ takes the position shown.	1. Any geometric figure may be moved about in space from one position to another without changing its size or shape.
2. Draw the radii OA, OB, OA', and OB'.	2. One and only one straight line can be drawn between two points.
3. $\widehat{AB} = \widehat{A'B'}$.	3. Given.
4. $\therefore \angle x = \angle x'$.	4. Radii which intercept equal arcs on the circle form equal central angles.

Proof (*Continued*)

STATEMENTS	REASONS
5. Also, $AO = A'O$, and $OB = OB'$.	5. Radii of the same circle or of congruent circles are equal.
6. $\therefore \triangle AOB \cong \triangle A'B'O$.	6. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
7. $\therefore AB = A'B'$.	7. Homologous parts of congruent triangles are equal.

EXERCISES

1. State and prove the converse of the preceding theorem.

2. If a circle is divided into three equal arcs and the points of division are joined by chords, the figure formed is an equilateral triangle.

3. What kind of a polygon is formed by joining the ends of a pair of perpendicular diameters of a circle? Prove your answer.

4. Show that a diameter bisects a circle.

5. Fig. 285 represents an equilateral *Gothic arch*, a form of arch much used in church architecture. Explain how the figure is constructed.

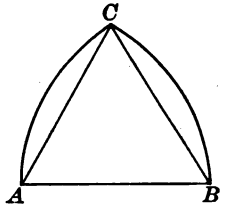


FIG. 285

6. Fig. 286 represents the design of a Gothic church window. $\overset{\frown}{AC}$ is constructed with B as a center and AB as a radius. $\overset{\frown}{BC}$ is constructed with A as a center and BA as a radius. The small arcs are drawn with radii equal to $\frac{1}{2}AB$ and with centers at A, at the midpoint of AB, and at B. The center of the $\odot O$ is found by constructing the two dotted arcs with A and B as centers and radii equal to $\frac{3}{4}AB$. Construct the figure.

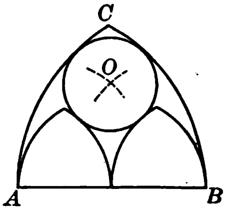


FIG. 286

300. Theorem. *A line through the center of a circle perpendicular to a chord bisects the chord and the arcs formed by the chord.*

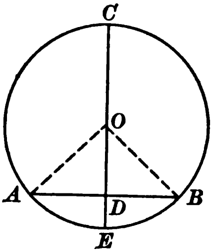


FIG. 287

Given the circle O, with CE perpendicular to the chord AB.

To prove that $AD = DB$, $\widehat{AE} = \widehat{EB}$, *and* $\widehat{AC} = \widehat{CB}$.

Proof

STATEMENTS	REASONS
1. Draw the radii AO and OB.	1. One and only one straight line can be drawn between two points.
2. Then $AO = OB$.	2. Radii of the same circle are equal.
3. $OD \equiv OD$.	3. Identical.
4. $CD \perp AB$.	4. Given.
5. $\therefore \triangle AOD \cong \triangle ODB$.	5. Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.
6. $\therefore AD = DB$.	6. Homologous parts of congruent triangles are equal.
7. $\angle x = \angle x'$.	7. Same reason as 6.
8. $\therefore \widehat{AE} = \widehat{EB}$.	8. Equal central angles have equal arcs.
9. $\widehat{EAC} = \widehat{EBC}$.	9. A diameter bisects a circle.
10. $\therefore \widehat{AC} = \widehat{CB}$.	10. Equals subtracted from equals give equals.

301. Corollary. *The perpendicular bisector of a chord of a circle passes through the center of the circle and bisects the arcs formed by the chord.*

EXERCISES

1. State and prove the propositions which can be formulated from the following data, using any two of the five conditions for the hypothesis and the remaining three for the conclusion:

(a) a line through the center of a circle;
(b) a perpendicular to a chord;
(c) a bisector of a chord;
(d) a bisector of a major arc;
(e) a bisector of a minor arc.

2. Draw a circle through two fixed points. How many such circles can be drawn?

3. Through a given point within a circle draw a chord which shall be bisected by the given point.

4. Draw a circle through three fixed points which are not collinear. How many such circles may be drawn? Prove your answer.

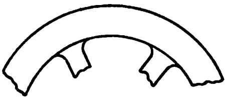


FIG. 288

5. Fig. 288 represents part of a broken wheel. Show how the radius of the original wheel may be found in order that a new wheel of the same radius may be made.

6. Through a point within a circle draw the longest possible chord.

7. Show by Art. 300 that the perpendicular bisectors of the sides of an inscribed polygon are concurrent.

8. Circumscribe a circle about a triangle.

9. Draw a circle that will pass through two given points and have a radius of given length.

SUGGESTION. Compare the given data with that given in the statement of the corollary of Art. 301.

302. Theorem. *In the same circle or in congruent circles equal chords are equidistant from the center.*

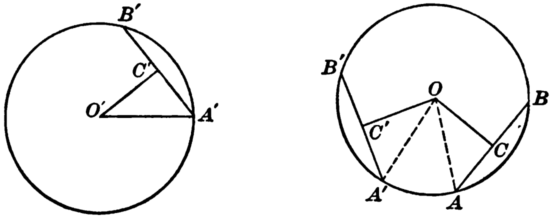


FIG. 289

Given the congruent circles O and O', with the chord AB equal to the chord $A'B'$.

To prove that $O'C' = OC = O'C$.

Proof

STATEMENTS	REASONS
1. Imagine the ⊙ O' superposed on the ⊙ O, as shown in the figure.	1. Any geometric figure may be moved about in space from one position to another without changing its size or shape.
2. Draw the radii OA and OA'.	2. One and only one straight line can be drawn between two points.
3. $OA = OA'$.	3. Radii of the same circle are equal.
4. $OC' \perp A'B'$, and $OC \perp AB$.	4. The distance from the center to the chord means the shortest, or perpendicular, distance.
5. $B'C' = C'A'$, and $AC = CB$.	5. A line through the center of a circle perpendicular to a chord bisects the chord.
6. But $A'B' = AB$.	6. Given.
7. $\therefore A'C' = AC$.	7. Halves of equals are equal.

Proof (*Continued*)

STATEMENTS	REASONS
8. $\therefore \triangle OC'A' \cong \triangle OAC$.	8. Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.
9. $\therefore OC' = OC$.	9. Homologous parts of congruent triangles are equal.
10. But $OC' = O'C'$.	10. Identical.
11. $\therefore OC' = OC = O'C'$.	11. Things equal to the same thing are equal to each other.

EXERCISES

1. State and prove the converse of the preceding theorem.

2. What is the locus of the midpoints of all equal chords of a circle ?

3. In Fig. 290 AC and AE are secants which make equal angles with AO. Prove that chord $BC =$ chord DE.

4. What is the locus of the upper outside corner of your classroom door as the door is opened ? What is the locus of the top edge of the door ? What kind of an angle is formed by two positions of the door ?

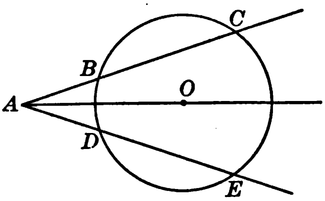


FIG. 290

5. What is the theoretical locus of all possible "hits" of a piece of artillery which is firing at a range of 8 mi.?

NOTE. In actual practice some shots will fall "short" and some will fall "over" the given range.

6. What is the locus of the centers of all parallel chords in a circle ?

7. If two intersecting chords make equal angles with the diameter through the point of intersection, the chords are equal.

303. ***Theorem.*** *A tangent to a circle is perpendicular to the radius at the point of contact.*

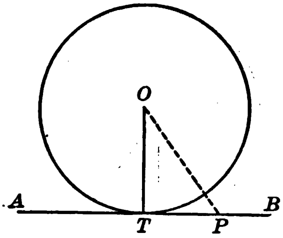


FIG. 291

Given the circle O, with AB tangent to the circle at T.

To prove that $OT \perp AB$.

Proof

STATEMENTS	REASONS
1. OT will be $\perp AB$ if we can show that it is the shortest distance from O to AB.	1. The shortest distance from a point to a line is measured on a perpendicular from the point to the line.
2. Take P, any other point except T on AB, and draw OP.	2. One and only one line can be drawn between two points.
3. Since AB is given as tangent to the circle, T is the only point where it touches the circle.	3. By the definition of a tangent.
4. $\therefore OP > OT$.	4. A point outside a circle is obviously farther away from the center than the end of a radius on the circle.
5. $\therefore OT \perp AB$.	5. It is shorter than OP, which represents any line other than OT.

EXERCISE

Through a given point within a circle draw two equal chords.

304. Theorem. *A line perpendicular to a radius at its extremity on the circle is tangent to the circle at that point.*

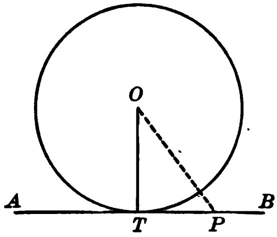


FIG. 292

Given the circle O, with AB perpendicular to OT at T.

To prove that *AB is tangent to the circle at T.*

Proof

STATEMENTS	REASONS
1. In order to prove that the line AB is tangent to the ⊙ O at T, we must show that it touches the circle only at the point of contact T.	1. A tangent is a line that touches a circle at only one point.
2. Take any point P on AB other than T. Then $OT < OP$.	2. A perpendicular is the shortest distance from a point to a line.
3. $\therefore$ since OP is any other line than OT, P is farther from O than T, and hence lies outside the circle.	3. Obvious.
4. $\therefore$ AB is tangent at T.	4. By definition of a tangent.

305. Corollary. *A perpendicular to a tangent to a circle at the point of contact passes through the center of the circle, and, conversely, a perpendicular from the center of a circle to a tangent passes through the point of contact.*

EXERCISES

1. Show how to draw a tangent to a circle at a given point on the circle. How many such tangents can be drawn?

2. In Fig. 293 we have a circle in which a large number of equal chords are drawn. Show why these chords seem to form a concentric circle.

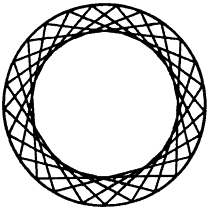
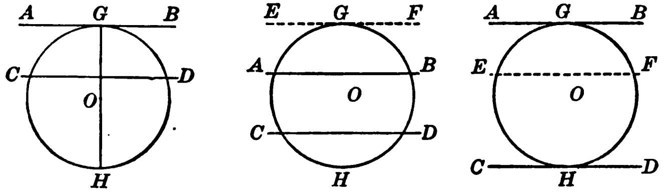


FIG. 293

306. *Theorem.* ***The arcs included between two parallel lines which cut a circle are equal.***



I II III

FIG. 294

Given the circle O cut by the two parallel lines AB and CD.

To prove that $\overset{\frown}{CG} = \overset{\frown}{GD}$ *and* $\overset{\frown}{CH} = \overset{\frown}{HD}$.

Proof

STATEMENTS	REASONS
There are three cases: CASE I. When one of the parallel lines is a tangent and one is a secant. CASE II. When both parallel lines are secants. CASE III. When both parallel lines are tangents.	
1. We shall take Case I first. Draw a diameter HG. Then $HG \perp AB$.	1. A tangent to a circle is perpendicular to a radius at its outer end.

Proof (*Continued*)

STATEMENTS	REASONS
2. $\therefore HG \perp CD$.	2. A line perpendicular to one of two parallel lines is perpendicular to the other also.
3. $\therefore \overset{\frown}{CG}=\overset{\frown}{GD}$, and $\overset{\frown}{CH}=\overset{\frown}{HD}$.	3. A line through the center of a circle perpendicular to a chord bisects the chord and its arc.
4. To prove Case II, draw the tangent $EF \parallel AB$. The proof is then the same as that of Case I.	
5. To prove Case III, draw the secant $EF \parallel AB$ and proceed as in Case I.	

EXERCISES

1. If two lines include equal arcs on a circle and do not intersect, they are parallel.

2. Show that Ex. 1 is the converse of the theorem of Art. 306.

307. Tangent circles. Two circles which are each tangent to the same line at the same point of the line are called *tangent circles.* When the two circles lie on the same side of the common tangent line, the circles are said to be *tangent internally*, and when they lie on opposite sides of the common tangent line, they are said to be *tangent externally.*

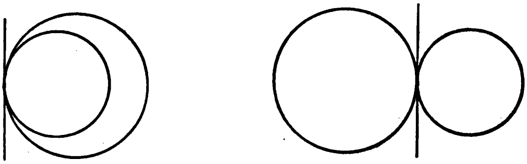


FIG. 295

EXERCISES

1. If two circles are tangent to each other, either externally or internally, their centers and the point of tangency are collinear (see Fig. 296).

2. If two circles are tangent externally, the distance between their centers is the sum of their radii.

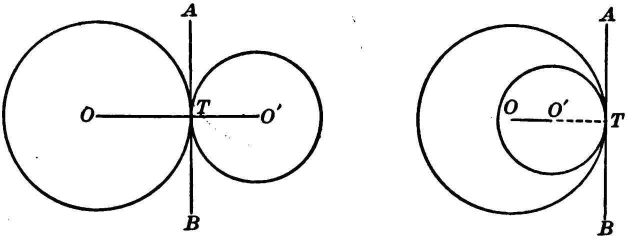


Fig. 296

3. State and prove the converse of Ex. 2.

4. If two circles are tangent internally, the distance between their centers is the difference of their radii.

5. Draw a circle of given radius tangent to a given circle both externally and internally. Discuss the various possibilities.

6. Draw a circle tangent to a given circle at a given point on the circle. How many such circles can be drawn?

7. Draw a line tangent to a given circle and parallel to a given line.

8. Draw a circle that shall pass through a given point and be tangent to a given line.

9. Draw a circle of given radius tangent to two given circles.

10. Show how to construct a circle which shall be tangent to three given lines, no two of which are parallel.

11. Draw a common tangent to two tangent circles.

12. The distance between the centers of two tangent circles is $3\frac{1}{2}$ in. The radius of one is $\frac{1}{2}$ in. Draw the two circles.

13. Three circles are tangent externally, as in Fig. 297. The lines of centers are $AC = 18$, $BC = 23$, and $AB = 30$. Find the radii of the three circles.

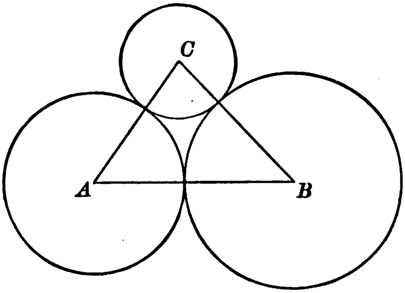


FIG. 297

14. The radii of three circles are $\frac{1}{2}$ in., $\frac{3}{4}$ in., and 1 in. respectively. Draw the circles so that they shall be tangent externally.

15. Draw a circle of given radius which shall pass through two given points.

308. Common chord. If two circles intersect in two points, the chord joining these points is called the *common chord* of the circles, as AB in Fig. 298.

EXERCISES

1. The line joining the centers of two intersecting circles is the perpendicular bisector of the common chord.

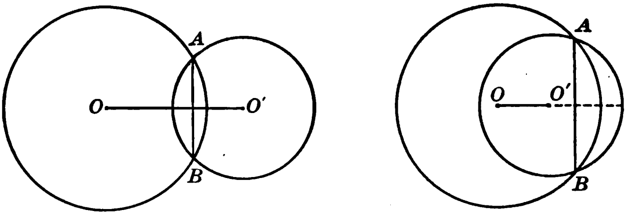
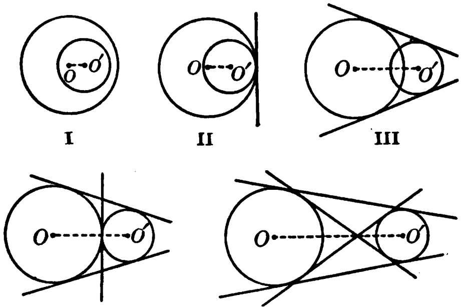


FIG. 298

2. A diameter which bisects a chord bisects the central angle between the radii from the ends of the chord.

3. The perpendicular bisectors of the sides of an inscribed quadrilateral are concurrent.

4. If two intersecting chords make equal angles with a line joining their point of intersection to the center of the circle, the chords are equal.



I II III

IV V

Fig. 299

5. Fig. 299 shows the various positions of two circles and the lines common to the circles. Discuss the various positions and explain what each figure illustrates.

6. Draw a $\triangle ABC$, and with A, B, and C as centers construct three circles tangent each to each (see Fig. 300).

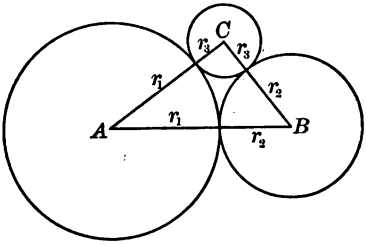


Fig. 300

7. Prove that if the ends A and D and the ends B and C of two parallel chords AB and DC of a circle are joined, $ABCD$ is an isosceles trapezoid.

8. Show that the diagonals of an isosceles trapezoid are equal.

309. ***Theorem.*** ***In the same circle or in congruent circles two central angles have the same ratio as the arcs intercepted by their sides.***

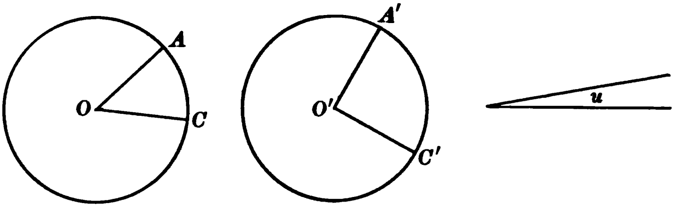


FIG. 301

Given the congruent circles O and O', with the central angles AOC and $A'O'C'$.

To prove that $$\frac{\angle AOC}{\angle A'O'C'} = \frac{\overset{\frown}{AC}}{\overset{\frown}{A'C'}}.$$

Proof

STATEMENTS	REASONS
1. Assume that the given angles are commensurable and that the angle unit u is the common unit of angle measure.	1. We shall here agree to consider only commensurable cases in this course (see note on page 227).
2. Suppose that u is contained m times in $\angle AOC$ and n times in $\angle A'O'C'$.	2. If the angles are commensurable, the unit must be contained an integral number of times in each angle.
3. Then $\angle AOC = mu$, and $\angle A'O'C' = nu$.	3. Obvious.
4. $\therefore \dfrac{\angle AOC}{\angle A'O'C'} = \dfrac{mu}{nu} = \dfrac{m}{n}$.	4. The ratio of the two angles is equal to the ratio of their numerical measures.
5. The sides of the unit $\angle u$ intercept m equal arcs (call each one y) on $\overset{\frown}{AC}$ and n equal arcs on $\overset{\frown}{A'C'}$.	5. In the same circle or in congruent circles equal central angles intercept equal arcs on the circle.

Proof (*Continued*)

STATEMENTS	REASONS
6. $\therefore \overset{\frown}{AC} = my,$ and $\overset{\frown}{A'C'} = ny.$	6. Obvious.
7. $\therefore \dfrac{\overset{\frown}{AC}}{\overset{\frown}{A'C'}} = \dfrac{my}{ny} = \dfrac{m}{n}.$	7. The ratio of the two arcs is equal to the ratio of their numerical measures.
8. $\therefore \dfrac{\angle ABC}{\angle A'O'C'} = \dfrac{\overset{\frown}{AC}}{\overset{\frown}{A'C'}}.$	8. Things equal to the same thing are equal to each other.

NOTE. When two geometric magnitudes each contain a certain unit of measure an integral number of times, this unit is called a *common unit of measure* and the magnitudes are said to be *commensurable*; otherwise they are said to be *incommensurable.* These cases will be considered in detail in Arts. 340 and 341. The theorem is also true for the case where the central angles are incommensurable, but we shall not prove that case in this course.

EXERCISES

1. Show by the theorem of Art. 309 that a central angle is measured by the arc intercepted by its sides.

SUGGESTION. Compare the given central angle with the perigon and the given arc with the entire circle.

2. By means of the theorem of Art. 309 divide a circle into two equal arcs; three equal arcs; four equal arcs.

3. Divide a circle into two arcs which have the ratio $\frac{2}{3}$.

4. By means of ruler and compasses only, construct arcs of 30°, 45°, 60°, 90°, 120°, and 150°.

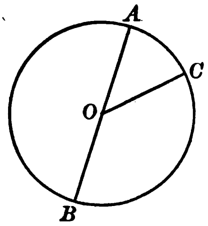


FIG. 302

5. In Fig. 302 $\angle AOC$ contains $x^2 + 2x$ degrees and $\angle BOC$ contains $9x^2 + 28x$ degrees. Find the number of degrees in $\overset{\frown}{AC}$ and in $\overset{\frown}{BC}$.

310. Intersecting lines and a circle. The parts of Fig. 303 are intended to show the student the various situations in which two intersecting lines may cut or touch a circle. We shall presently have a theorem connected with each part of the figure except I, which shows two lines intersecting at the center of a circle. We discussed this situation in connection with the problem of Art. 41.

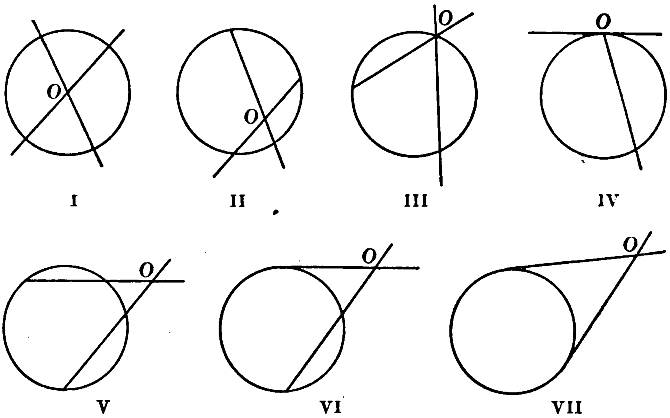


FIG. 303

The student should notice that in each case the angles between the two lines can obviously be measured by certain combinations of the arcs cut by the lines on the circle.

The student should also observe that as the intersection of the lines moves from the center of the circle, as shown in I, to the position shown in II, thence to the position on the circle shown in III and IV, and thence to the position outside the circle, as shown in V, VI, and VII, the measures of the angles change. These changes can be easily discovered and the entire situation clearly understood if the student will imagine the measures as continuously changing from one position of the cutting lines to the other and will observe all the accompanying changes in the intercepted arcs.

The parts of Fig. 303 illustrate what is known as the *Principle of Continuity.*

In mathematics if a certain theorem or truth holds true for all possible cases under consideration, we have an illustration of the Principle of Continuity. If the student will make a careful study of the theorems connected with the various figures in Fig. 303 he will get a good idea of this important principle.

We shall refer to the angles formed by the two lines in Fig. 303 as follows:

In I the two lines intersect inside the circle at the center. The angle is formed, therefore, by two radii, and is called *a central angle.*

In II the two lines intersect inside the circle but not at the center, and the angle is called *an angle formed by two chords.* This case is covered by Ex. 3, page 235.

In III the lines intersect on the circle, and the angle is called an *inscribed angle* (see Art. 311).

In IV the lines also intersect on the circle, but one of the lines is a tangent. The angle is called *an angle formed by a tangent and a chord* (see Art. 313).

In V the lines intersect outside the circle, and the angle is called *an angle formed by two secants* (see Art. 314).

In VI the lines intersect outside the circle, one of the lines is a tangent, and the other is a secant. The angle is called *the angle formed by a secant and a tangent* (see Art. 315).

In VII the lines again intersect outside the circle, but both lines are tangents. The angle is called *an angle formed by two tangents* (see Art. 316).

311. *Theorem.* *An inscribed angle is measured by one half the arc intercepted by its sides.*

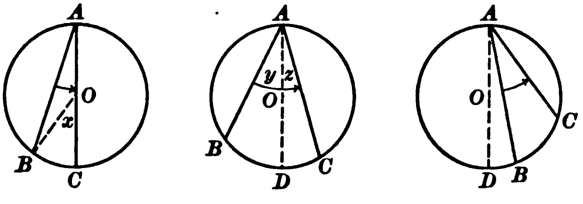


FIG. 304

Given the inscribed angle *BAC*.

To prove that *$\angle BAC$ is measured by $\frac{1}{2}\ \overset{\frown}{BC}$.*

Proof

STATEMENTS	REASONS
There are three cases to be considered, which we shall treat in order.	
CASE I. When one side of the inscribed angle is a diameter (Fig. 304, I).	
1. Draw OB.	1. One and only one line can be drawn between two points.
2. Then $\angle x = \angle A + \angle B$.	2. The exterior angle of a triangle is equal to the sum of the two remote interior angles.
3. But $OA = OB$.	3. Radii of the same circle are equal.
4. $\therefore \angle A = \angle B$.	4. Base angles of an isosceles triangle are equal.
5. $\therefore \angle x = 2\angle A$.	5. By substitution.
6. Or $\angle A = \frac{\angle x}{2}$.	6. By dividing both members of the equation by 2.
7. But $\angle x$ is measured by $\overset{\frown}{BC}$.	7. A central angle is measured by the arc intercepted by its sides.

Proof (*Continued*)

STATEMENTS	REASONS
8. $\therefore \angle A$ is measured by $\frac{1}{2}\overset{\frown}{BC}$.	8. By substitution.
CASE II. When the center of the circle lies within the angle.	
9. Draw the diameter AD.	9. A diameter of a circle may be drawn.
10. $\angle y$ is measured by $\frac{1}{2}\overset{\frown}{BD}$.	10. By Case I, above.
11. $\angle z$ is measured by $\frac{1}{2}\overset{\frown}{DC}$.	11. By Case I, above.
12. $\therefore \angle y + \angle z$ is measured by $\frac{1}{2}\overset{\frown}{BDC}$.	12. By the addition of 10 and 11.
13. $\therefore \angle BAC$ is measured by $\frac{1}{2}\overset{\frown}{BC}$.	13. The whole is equal to the sum of its parts.
CASE III. When the center of the circle is outside the angle.	
14. $\angle DAC$ is measured by $\frac{1}{2}\overset{\frown}{DBC}$.	14. By Case I, above.
15. $\angle DAB$ is measured by $\frac{1}{2}\overset{\frown}{DB}$.	15. By Case I, above.
16. $\therefore \angle BAC$ is measured by $\frac{1}{2}\overset{\frown}{BC}$.	16. By subtracting 15 from 14.

312. Corollary 1. *An angle inscribed in a semicircle is a right angle.*

Corollary 2. *All inscribed angles intercepting the same arc of a circle are equal.*

EXERCISES

1. An inscribed angle intercepting an arc greater than a semicircle is obtuse.

2. An inscribed angle intercepting an arc less than a semicircle is acute.

3. If an inscribed angle intercepted by it is a right angle, the arc is a semicircle.

4. If an inscribed angle is obtuse, the arc intercepted by it is greater than a semicircle.

5. If an inscribed angle is acute, the arc intercepted by it is less than a semicircle.

6. How does the size of an inscribed angle vary as the intercepted arc increases from 0° to 360°?

7. If two chords intersect within a circle and the end points are joined by chords, the angles of the triangles thus formed are respectively equal.

8. In fitting a board around a pipe a carpenter cuts out a semicircular section, as shown in Fig. 305. Show how he may use his square to determine whether or not he has actually cut out a semicircle.

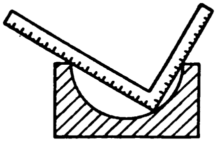


FIG. 305

9. Show how a carpenter's square may be used to divide a ring (Fig. 306) into two equal parts.

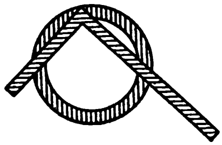


FIG. 306

10. In Fig. 307 AB is a given line segment, and it is required to erect a perpendicular at A. If with any point O, not on AB, as center and OA as radius a part of a circle is drawn cutting AB at C, and if CO is produced to intersect the circle at D, then the line joining D and A is the required perpendicular. Prove that the construction is correct.

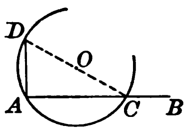


FIG. 307

11. The bisector of an inscribed angle bisects the intercepted arc.

12. The bisectors of all inscribed angles which intercept the same arc of a circle are concurrent.

13. The opposite angles of an inscribed quadrilateral are supplementary.

14. If the opposite angles of a quadrilateral are supplementary, a circle may be circumscribed about the quadrilateral.

SUGGESTION. Construct a circle (Fig. 308) passing through three of the vertices, *A*, *D*, and *C*, and show by the indirect method that the fourth vertex lies on the circle; that is, show that the circle cannot cut *CB* at *E* nor *CB* produced at *F*.

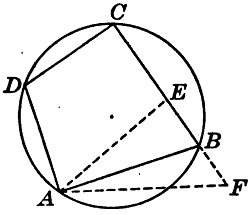


FIG. 308

15. Show by means of the theorem of Art. 311 that the sum of the interior angles of the $\triangle ABC$ (Fig. 309) is a straight angle.

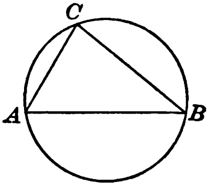


FIG. 309

16. If, as shown in Fig. 310, three angles have their vertices inside the circle, on the circle, and outside the circle respectively, show that the one with its vertex inside will be greater than the inscribed angle, and the one with its vertex outside will be less than the inscribed angle.

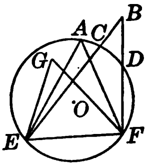


FIG. 310

17. If two circles intersect and a secant cutting both circles is drawn through each of the points of intersection, the two chords joining the corresponding intersections of the secants and the circles are parallel.

18. If two secants are drawn through the point of contact of two internally tangent circles (Fig. 311), the two chords connecting the corresponding intersections of the secants and the circles are parallel.

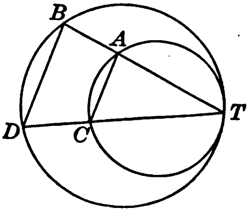


FIG. 311

19. If from either point of intersection of two intersecting circles the diameters of the two circles are drawn, then the other point of intersection lies on the straight line connecting the other ends of the diameters.

20. Prove that parallel chords, drawn from the extremities of a diameter of a circle, are equal.

21. Prove that the lines joining the ends of two intersecting equal chords of a circle form an inscribed isosceles trapezoid.

22. Prove that the bisectors of the angles of a circumscribed quadrilateral are concurrent.

23. In avoiding rocks and shoals near a coast, mariners sometimes depend upon what is known as the *horizontal danger angle*, which is used in the following manner: A and C (Fig. 312) are two lighthouses on shore, and the danger angle ($\angle ABC$) is the angle inscribed in a circle passing through A and C and of sufficient radius to inclose the danger zone. The ship must then be kept in such a position that at all times the angle between the lighthouses, as observed from the ship, is less than the known danger angle (usually given on the chart). Explain why this method is effective.

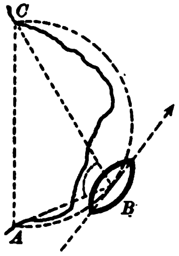


FIG. 312

313. Theorem. *An angle formed by a tangent and a chord which passes through the point of contact is measured by one half the intercepted arc.*

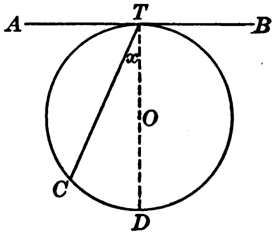


FIG. 313

Given the circle O, with the tangent AB and the chord CT.

To prove that $\angle ATC$ *is measured by* $\frac{1}{2}\widehat{CT}$

and $\angle BTC$ *is measured by* $\frac{1}{2}\widehat{CDT}$.

Proof

STATEMENTS	REASONS
1. Draw the diameter TD.	1. A diameter of a circle may be drawn.
2. Then $\angle ATD = 90°$.	2. A line drawn through the center of a circle to the point of tangency is perpendicular to the tangent.
3. $\therefore \angle ATD$ is measured by $\frac{1}{2}\overset{\frown}{TCD}$.	3. A diameter divides the circle into two arcs of 180° each.
4. But $\angle x$ is measured by $\frac{1}{2}\overset{\frown}{CD}$.	4. An inscribed angle is measured by one half the arc intercepted by its sides.
5. $\therefore \angle ATC$ is measured by $\frac{1}{2}\overset{\frown}{TC}$.	5. By subtraction.
6. Likewise, $\angle BTD = 90°$.	6. A line drawn through the center of a circle to the point of tangency is perpendicular to the tangent.
7. $\therefore \angle BTD$ is measured by $\frac{1}{2}\overset{\frown}{DT}$.	7. Since DT is a diameter, the arc TD is an arc of 180°.
8. $\angle x$ is measured by $\frac{1}{2}\overset{\frown}{CD}$.	8. Shown above.
9. $\therefore \angle BTC$ is measured by $\frac{1}{2}\overset{\frown}{CDT}$.	9. By addition.

EXERCISES

1. Prove the preceding theorem by drawing a line through C parallel to AB.

2. Prove the preceding theorem by drawing a line from O perpendicular to CT.

3. If two chords AE and DC intersect at a point B inside a circle, either angle they form is measured by *one half the sum* of the intercepted arcs.

SUGGESTION. Draw EC and prove that $\angle ABC = \angle E + \angle C$. Then give the complete proof.

314. *Theorem.* *An angle formed by two secants which intersect outside a circle is measured by one half the difference of the intercepted arcs.*

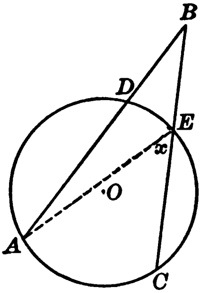


FIG. 314

Given the circle O and the secants AB and BC intersecting outside the circle.

To prove that *$\angle B$ is measured by $\frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{DE})$.*

Proof

STATEMENTS	REASONS
1. Draw AE.	1. One and only one straight line can be drawn between two points.
2. Then $\angle x = \angle B + \angle A$.	2. An exterior angle of a triangle is equal to the sum of the two remote interior angles.
3. $\therefore \angle B = \angle x - \angle A$.	3. Subtracting $\angle A$ from each member of the equation.
4. But $\angle x$ is measured by $\frac{1}{2}\overset{\frown}{AC}$, and $\angle A$ is measured by $\frac{1}{2}\overset{\frown}{DE}$.	4. An inscribed angle is measured by one half the arc intercepted by its sides.
5. $\therefore \angle B$ is measured by $\frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{DE})$.	5. By substitution.

NOTE. The student should try to see if he can show the similarity of the preceding theorem to those of Arts. 311 and 313.

315. Theorem. *An angle formed by a secant and a tangent which intersect outside a circle is measured by one half the difference of the intercepted arcs.*

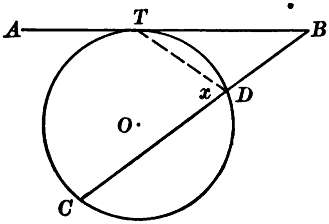


FIG. 315

316. Theorem. *An angle formed by two tangents which intersect outside a circle is measured by one half the difference of the intercepted arcs.*

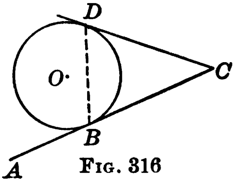


FIG. 316

Proof. In these two theorems the student should supply the proof, which is similar to that of the theorem of Art. 314.

EXERCISES

1. Using Fig. 303, show that in each case, I to VII inclusive, the measures of the angles formed by the two intersecting lines can be found by the formula

$$A = \frac{x+y}{2},$$

where A is the angle and x and y are the intercepted arcs. Show how this illustrates the Principle of Continuity.

SUGGESTION. Notice that when the point of intersection of the lines moves to the circle, then $y = 0$; and when the point moves outside the circle, then y is negative.

2. In Fig. 317 $\triangle A'B'C'$ is inscribed in a circle, $\angle C' = 47°$, $\angle A' = 56°$, and AB, BC, and CA are tangent to the circle at C', A', and B' respectively. Find the size of $\angle A$, $\angle B$, and $\angle C$.

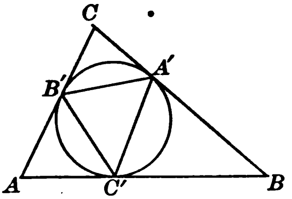


Fig. 317

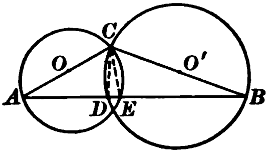


Fig. 318

3. If, in Fig. 317, $\angle A = 70°$ and $\angle B = 30°$, find the size of $\angle A'$, $\angle B'$, and $\angle C'$.

4. See if you can discover and explain the fallacy in the proof of the following theorem: ***From a given point outside a given line two perpendiculars can be drawn to the line.***

Proof

Statements	Reasons
1. Draw the two circles as shown in Fig. 318 and draw the diameters CA and CB.	1. A diameter may be drawn in a circle.
2. Draw AB meeting the two circles in D and E respectively.	2. One and only one straight line may be drawn between two points.
3. Then $\angle CEA = 90°$, and $\angle CDB = 90°$.	3. An angle inscribed in a semicircle is a right angle.
4. $\therefore CD \perp AB$, and $CE \perp AB$.	4. By definition of a perpendicular.
5. ∴ two perpendiculars can be drawn from a point outside a line to that line.	

Note. The explanation of the fallacy in the preceding exercise will show the student the need for accurately drawn figures and carefully worked out proofs. Other fallacies and paradoxes will be given later.

5. In Fig. 319 AC and CB are diameters, and C and D are the intersection points of the circles. Prove that ADB is a straight line.

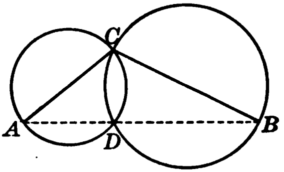


Fig. 319

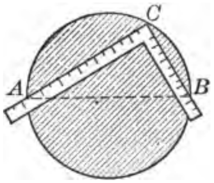


Fig. 320

6. Fig. 320 shows a circular object whose center we wish to find. If we lay a carpenter's steel square on the object, with the vertex of the right angle (C) on the circle, and mark the points A and B where the legs cut the circle, then AB is a diameter. Show why this is so, and show how to find the radius of the object.

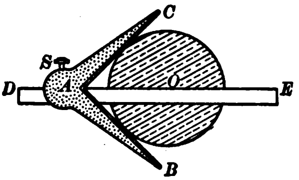


Fig. 321

7. A *center square* (Fig. 321) is another instrument which is used for locating the center of a circular object. The head A, carrying the two arms B and C, slides upon the steel blade DE and can be fixed at any point on the blade by the screw S. The upper edge of the blade DE always bisects the $\angle BAC$. If the center square is placed so that AB and AC are tangent to a circular object, then AE passes through the center of the object. Explain why this is true.

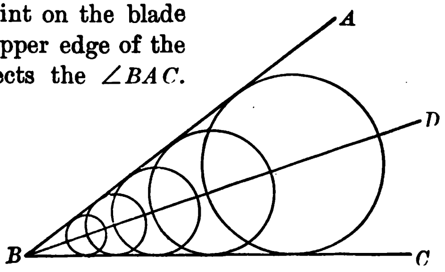


Fig. 322

8. Show that the locus of the centers of all circles which are tangent to the sides of the $\angle ABC$ in Fig. 322 is the bisector BD of the $\angle ABC$.

9. In a right triangle circumscribed about a circle prove that the diameter of the circle is equal to the difference between the hypotenuse and the other two sides.

10. Prove that an inscribed angle is greater than the angle formed by two secants which intercept the same arc.

11. The radius of the circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.

317. Summary. The following table summarizes the theorems concerning angles formed by two intersecting lines which cut a circle and shows the classification of the angles, their measures, and the location of their vertices:

THE ANGLES ARE FORMED BY	THE ANGLES ARE MEASURED BY	THE VERTEX OF THE ANGLES IS
1. Two chords.	1. One half the sum of the intercepted arcs.	1. Inside the circle.
2. Two chords which intersect at the center (central angle).	2. The intercepted arcs.	2. Inside the circle.
3. Two chords which intersect on the circle (inscribed angle).	3. One half the intercepted arc.	3. On the circle.
4. Two secants.	4. One half the difference of the intercepted arcs.	4. Outside the circle.
5. A tangent and a secant.	5. One half the difference of the intercepted arcs.	5. Outside the circle.
6. Two tangents.	6. One half the difference of the intercepted arcs.	6. Outside the circle.

318. Geometric constructions. The theorems which we have studied and the principles of loci which we have discussed furnish a basis for making certain fundamental geometric constructions. In making these constructions it is not always necessary to know all the parts of the figure to be constructed. For example, a triangle is said to be *determined* when two sides and the angle included between these two sides are given, because if these three parts are known the other three parts are easily determined.

The three sides and the three angles of a triangle are the *principal parts* of the triangle.

It often happens that a triangle may be determined by knowing certain parts not all of which are principal parts, such as the altitudes, the bisectors of the interior angles, the medians, the radii of the inscribed and circumscribed circles, and so on. Such parts are called *secondary parts* of the triangle.

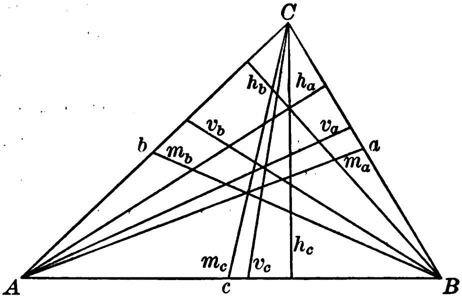


FIG. 323

Fig. 323 shows a scheme of notation for secondary parts of a triangle. Thus, in the figure h_a, h_b, and h_c represent the altitudes to the sides a, b, and c respectively; m_a, m_b, and m_c represent the medians to the sides a, b, and c respectively; and v_a, v_b, and v_c, the bisectors of the angles A, B, and C respectively. We usually let r and R represent the radii of the inscribed and circumscribed circles respectively, and let r_a, r_b, and r_c represent the radii of the *escribed circles*, which are the circles touching a, b, and c produced and having as centers the intersections of the bisectors of the exterior angles of the triangle.

319. Necessary and sufficient conditions for determining a triangle. We have seen that, in general, three parts are necessary and sufficient to determine a triangle, provided that one of the parts is a line, for it is evident that three angles do not determine a triangle. It is also obvious that we must make a choice of parts in such a way that there exists the proper relation between them; otherwise, we do not obtain the required triangle. For example, three sides do not determine a triangle if the sum of two of these sides is less than the third, nor can we construct a triangle when two angles and an included side are given if the sum of the two given angles is greater than a straight angle.

320. Determining a triangle. We have seen from our work on congruent triangles that a triangle is determined (that is, the triangle is fixed, and there is only one solution) when the following parts are given:

1. Two sides and the included angle.
2. A side and two adjacent angles.
3. Three sides.
4. In a right triangle, the hypotenuse, the right angle, and any other principal part.

In assuming the given parts, however, they must be chosen so that the proper relation exists.

EXERCISES

1. Discuss the limits within which each of the preceding constructions is possible.

2. What principal parts determine an isosceles triangle? an equilateral triangle?

321. Synthetic method in geometric constructions. The synthetic method (Art. 139) requires the direct application of known conditions or properties and therefore, by its very nature, is not generally applicable to a broad field of problems. It may be used, however, in such cases as the following: If we are asked to draw a tangent to a circle at a given point on the circle, we can do it directly, because we know that a tangent to a circle is perpendicular to a radius at the point of contact. Hence we may draw the radius from the center to the given point and then draw a line perpendicular to the radius at the given point. This perpendicular will then be the required tangent (see Art. 326).

When the application of the synthetic method is obvious, it is advisable to use it; but we shall have many problems where a different method will be more efficient.

322. Analytic method in geometric constructions. The analytic method, or method of analysis, is very powerful when applied to the construction of geometric figures, and the student should learn to use it freely in all construction work. This method enables one to discover the known relation upon which the construction depends. It consists largely in the following steps:

1. Read the problem carefully to get the requirements clearly in mind.
2. Draw a freehand figure which meets the given conditions.
3. Mark the parts which are known with colored crayons (if available), and indicate those parts which can be determined easily.
4. Look for a part of the figure (often a triangle) that can be constructed by known methods (often the principles of loci are helpful).
5. Determine the location of the remaining parts, drawing any helping lines that may be necessary.

6. Prove the construction correct.

7. Discuss all the possibilities of the figure, and show whether or not the construction is always possible.

NOTE. If the analysis is complete, a proof is unnecessary, as will be seen in the problem which follows.

323. ***Construction problem.*** *To construct a triangle when a side and the altitude and median to the side are given.*

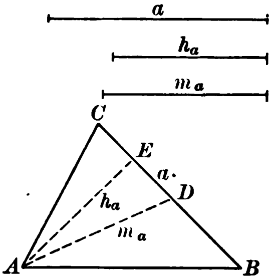


FIG. 324

Given the side a, the altitude h_a, and the median m_a.

Required *to construct the △ABC.*

Analysis. Imagine the triangle constructed, as shown in Fig. 324. A rt. △ADE can be constructed, because h_a will be a leg and m_a will be the hypotenuse. This leaves only the points B and C to be determined. We can see from the given parts that B and C are each at a distance of $\frac{1}{2}a$ from D. Hence the △ABC can be constructed as required.

Construction. Construct the rt. △ADE, using m_a as the hypotenuse and h_a as a leg. Produce DE in both directions, and on DE from D lay off DB and DC, each equal to $\frac{1}{2}a$. Draw AB and BC, and the △ABC is constructed as required.

Proof. Why is a proof unnecessary?

Discussion. This construction is not possible if $m_a < h_a$. Why?

324. Use of loci in geometric constructions. We can often use the principles of loci in determining the method of making certain constructions. This is especially true where the given conditions are such that fundamental points in the required figure are intersections of loci. Often the problem indicates that a point shall satisfy each of two given conditions. Each of these conditions determines a locus for the point. The point then must obviously lie at the intersection of the two loci. The use of this method may be seen in the problem which follows.

325. ***Construction problem.*** *To construct a circle of given radius tangent to the sides of a given angle.*

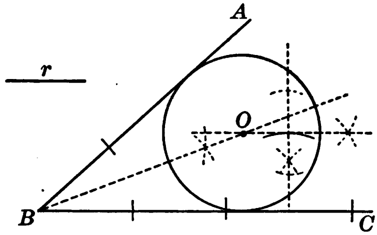


FIG. 325

Given the angle *ABC* and the radius *r*.

Required *to construct a circle of radius r tangent to AB and BC.*

Analysis. If r is the given radius and ABC is the given angle, one locus of a possible center for the required circle is the bisector of the $\angle ABC$, and another locus is a line $\parallel BC$ on the same side as AB and at a distance r from BC. These loci are obvious, since the circle must be tangent to AB and BC and the center must be at a distance r from each of AB and BC.

Construction. The complete construction and proof is left to the student.

326. *Construction problem.* *Through a given point, to draw a tangent to a given circle.*

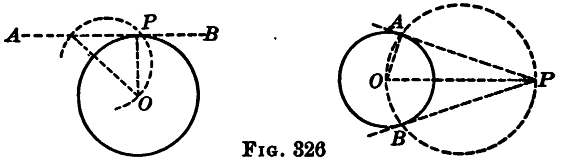


FIG. 326

Given the circle O and the point P.

Required *to construct a tangent to the circle O through the point P.*

There are two possibilities, which we shall treat in the following order:

CASE I. The given point P is on the circle.

Analysis. When the given point P is on the circle the problem is simply that of constructing a line AB through $P \perp OP$.

Construction. We construct a line $AB \perp OP$ at P by the method of Ex. 10, Art. 312. Then AB is the required tangent.

Proof. The proof is left to the student.

CASE II. The given point P is outside the circle.

Analysis. Imagine the lines PA and PB drawn from P to the $\odot O$. If the lines are to be tangent, they must be perpendicular to the radii OA and OB respectively. Hence the problem is to draw PA and PB so that they will form right angles with radii from O. This can be done if we know Corollary 1, Art. 312.

Construction. Connect O and P. With OP as a diameter describe a circle cutting the given circle at A and B. Draw PA and PB, which are the required tangents.

Proof. The proof is left to the student.

EXERCISES

1. Two tangents to a circle from a point outside the circle are equal.

2. What is the locus of the centers of all circles of given radius and tangent to a circle of given radius?

3. In Fig. 327 a circle is inscribed in the rt. $\triangle ABC$. Show that the sum of the two legs is equal to the sum of the hypotenuse and the diameter of the circle.

4. What is the locus of the centers of all circles with given radius and tangent to a given straight line?

5. Find the locus of all points that are equidistant from two given points and are also equidistant from two given parallel lines.

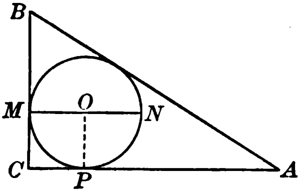


FIG. 327

6. Find the locus of all points which are equidistant from two given parallel lines and are also equidistant from two given intersecting lines.

327. *Construction problem.* *Upon a given line segment as a chord to construct an arc of a circle intercepted by an inscribed angle equal to a given angle.*

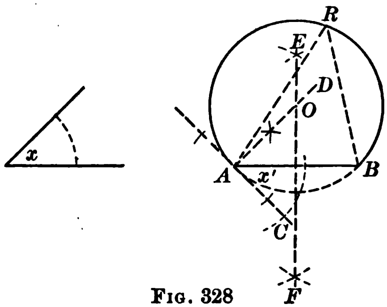


FIG. 328

Given the angle x and the line segment AB.

Required *to construct on AB as a chord an arc of a circle intercepted by an inscribed angle equal to the $\angle x$.*

Construction. If the line segment AB is to be a chord of the circle, it is evident that the center of the circle lies somewhere on the perpendicular bisector of AB, which is the locus of all points equidistant from A and B. Hence we draw EF as the perpendicular bisector of AB. Then, if we make $\angle x' = \angle x$, we know that the perpendicular at A will contain the centers of all circles to which AC is tangent, because if a line is perpendicular to a tangent at the point of contact it passes through the center of the circle.

Therefore we draw $AD \perp AC$ at A. Then AD intersects EF at O, because AD and EF are $\perp$ two intersecting lines and hence intersect. Then, using O as a center and OA as radius we draw the required circle.

Proof

STATEMENTS	REASONS
1. Draw any inscribed $\angle ARB$ intercepting $\widehat{ARB}$.	1. Any angle may be inscribed in a circle.
2. Then $\angle ARB$ is measured by $\frac{1}{2}\widehat{AB}$.	2. An inscribed angle is measured by one half the intercepted arc.
3. Also $\angle x'$ is measured by $\frac{1}{2}\widehat{AB}$.	3. An angle formed by a tangent and a chord is measured by one half the intercepted arc.
4. But $\angle x' = \angle x$.	4. By construction.
5. $\therefore \angle x$ is measured by $\frac{1}{2}\widehat{AB}$.	5. Things equal to the same thing are equal to each other.
6. $\therefore \angle x = \angle ARB$.	6. Same reason.

EXERCISES

1. Solve the preceding construction problem with a given obtuse angle.

2. Construct a $\triangle ABC$ when the base c, the altitude on c, and the angle opposite the base ($\angle C$) are given.

3. Construct a $\triangle ABC$ when c, C, and the median to c are given.

4. Construct a $\triangle ABC$ when c, the altitude on c, and the median to c are given.

5. What is the locus of the vertices of the right angles of all right triangles which have the same hypotenuse?

6. What is the locus of all points equidistant from two given intersecting lines and also equidistant from a given point?

7. Locate a point which shall be 1 cm. from the vertex of a given angle and equidistant from its sides.

8. What is the locus of the center of a circle which rolls around a square, always remaining coplanar with the square?

9. Construct three equal semicircles each of which shall be tangent to the other two.

10. Construct four equal circles in a square such that each circle is tangent to two of the others. (There are two solutions.)

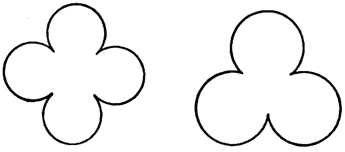


FIG. 329

11. The curve shown in Fig. 329, I, is called the *quatrefoil* and that shown in Fig. 329, II, is called the *trefoil*. Show how the curves are made.

328. Number of solutions. Construction problems sometimes have no solution and sometimes an indefinite number of solutions. If there is no solution, the problem is said to be *impossible*. For example, if we are asked to draw a tangent to a circle from a point inside the circle, the construction is obviously impossible. If there is a definite number of solutions (say one, two, or three), the problem is said to be *determinate*. If there is an indefinite number of solutions the problem is said to be *indeterminate*. In making constructions it is very important for the student to remember these facts when he is attempting to solve a particular problem.

EXERCISES

1. To construct an isosceles triangle when one of the equal sides and an altitude upon it are given.

2. To construct an isosceles triangle when the base and the vertex angle are given.

3. To construct an isosceles triangle when one base angle and the altitude to the base are given.

4. To construct a $\triangle ABC$ when A, v_a, and h_a are given.

5. To construct a $\triangle ABC$ when $\angle A$, h_b, and R, the radius of the circumscribed circle, are given.

6. To construct a triangle when one side, an angle adjacent to that side, and the difference of the other two sides are given.

7. To construct a right triangle having given the hypotenuse and one leg.

8. To construct a $\triangle ABC$ when the medians m_a and m_b and the altitude h_b are given.

9. To construct an equilateral triangle when the altitude is given.

10. To construct an isosceles triangle having given the following parts:

(a) The base and altitude.

(b) The base and one base angle.

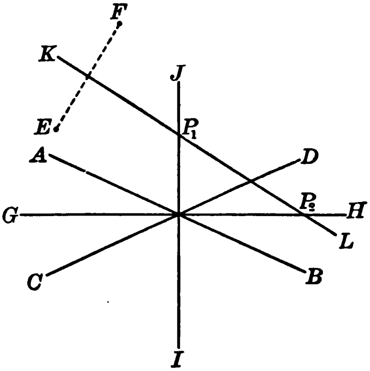


FIG. 330

11. To construct an isosceles triangle when the perimeter and the altitude are given; when the perimeter and the sum of the base angles are given.

12. Find the locus of points which are equidistant from two given intersecting lines and which are also equidistant from two given points (see Fig. 330).

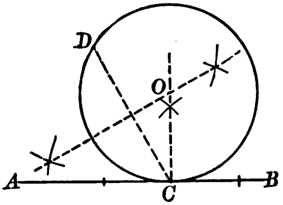


FIG. 331

13. To construct a circle tangent to a given line at a given point and passing through a given point not on the given line (see Fig. 331).

14. To construct a right triangle when the hypotenuse and the length of the altitude upon it are given (see Fig. 332).

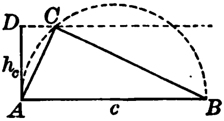


FIG. 332

15. To construct a right triangle when one leg and the altitude upon the hypotenuse are given (see Fig. 333).

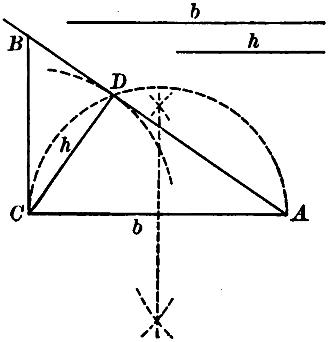


FIG. 333

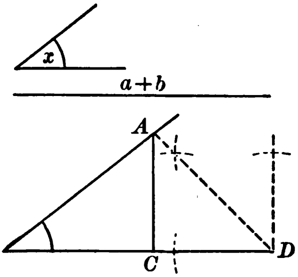


FIG. 334

16. To construct a right triangle when one of the acute angles and the sum of the two legs are given.

SUGGESTION. Let $\angle x$ (Fig. 334) be the given $\angle B$ and let $a + b$ be the given sum of the two legs. By analysis the student will see that $\angle ADC = 45°$.

17. To contruct a right triangle having given the following parts:

(a) One leg and the hypotenuse.
(b) The hypotenuse and the difference of the legs.
(c) The hypotenuse and the sum of the legs.
(d) An acute angle and the altitude upon the hypotenuse.

18. Construct a $\triangle ABC$ having given the following parts:

(a) a, b, and h_b.
(b) a, B, and h_b.
(c) b, c, and m_c.
(d) a, b, and h_c.
(e) a, h_c, and A.
(f) B, b, and m_b.
(g) b, h_b, and m_b.
(h) a, m_a, and B.
(i) m_a, h_a, and B.
(j) a, h_a, and h_c.
(k) h_a, h_b, and B.
(l) a, b, and $A + B$.

19. To construct a common external tangent to two given circles (see Fig. 335).

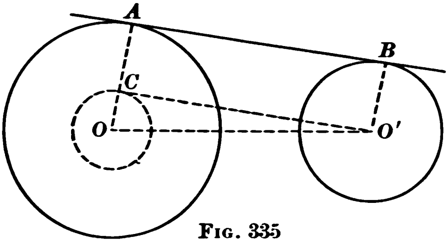


Fig. 335

20. To construct a common internal tangent to two given circles (see Fig. 336).

21. How many common tangents are there to two circles which have no point in common? Explain the two possibilities.

22. How many common tangents are there to two circles which are tangent internally? which are tangent externally? which intersect in two points?

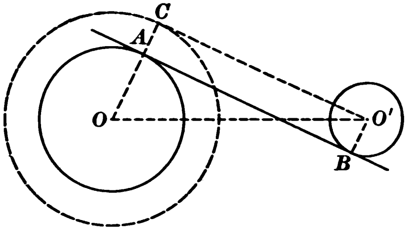


Fig. 336

329. Lunar eclipse. When the moon passes through the shadow of the earth there occurs a phenomenon known as a *lunar eclipse.*

If the moon (Fig. 337) is entirely within the dark part of the earth's shadow (the umbra) the eclipse is said to be *total.* The *umbra* is the part of the earth's shadow

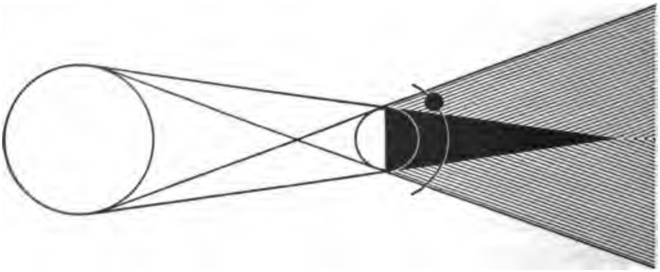


Fig. 337

which is included between the earth and the two common external tangents to the earth and sun.

If the moon is in the half-light portion of the earth's shadow (the penumbra) the eclipse is said to be *partial.* The limits of the *penumbra* are determined by the common internal tangents to the earth and sun.

330. ***Theorem.*** *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*

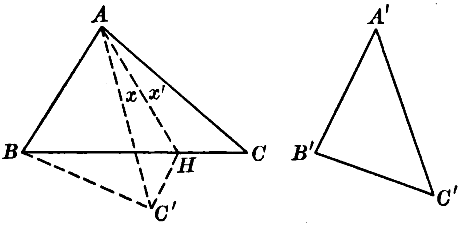


Fig. 338

Given the triangles ABC and $A'B'C'$, with AB equal to $A'B'$, AC equal to $A'C'$, and angle A greater than angle A'.

To prove that $BC > B'C'$.

Proof

Statements	Reasons
1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'B'$ coincides with its equal AB and so that C' falls to the right of AB, as shown in Fig. 338.	1. Any geometric figure may be moved about in space from one position to another without changing its size or shape.
2. Then $A'C'$ falls to the left of AC in the position AC'.	2. Because $\angle A$ was given greater than $\angle A'$.
3. C' will fall either above, on, or below BC.	3. Obvious.

Proof (*Continued*)

STATEMENTS	REASONS
4. We shall prove the theorem when C' falls below BC. If C' falls on BC, it is evident that the theorem is true. If C' falls above BC, the proof is similar to that which follows.	4. No reason necessary.
5. Draw AH bisecting the $\angle C'AC$.	5. An angle may be bisected.
6. In the $\triangle$s $AC'H$ and AHC, $AC' = AC$.	6. Given.
7. $\angle x = \angle x'$.	7. By construction.
8. $AH \equiv AH$.	8. Identical.
9. $\therefore \triangle AC'H \cong \triangle AHC$.	9. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
10. $\therefore C'H = HC$.	10. Homologous parts of congruent triangles are equal.
11. In the $\triangle BC'H$, $BH + HC' > BC'$.	11. The sum of two sides of a triangle is greater than the third side.
12. $\therefore BH + HC > BC'$.	12. By substituting HC for HC'.
13. $\therefore BC > BC'$.	13. By substituting BC for $BH + HC$.

NOTE. The preceding theorem is frequently applied in mechanics and physics, and in such cases is generally stated somewhat as follows: *As an angle of a triangle is allowed to increase, the side opposite increases accordingly.* It is understood, of course, that the other two sides remain constant. An illustration of this is given in Ex. 1, page 256. There are many other interesting applications of geometric theorems and principles about us in our daily lives. The student should be on the lookout for such illustrations, and should bring to class any such applications as he may find.

EXERCISES

1. Discuss the case in the preceding theorem where C' falls on BC; where C' falls inside the $\triangle ABC$.

2. What happens to the third side of each of the triangles in the preceding theorem if the angle included between the given sides is increased?

3. How does the preceding theorem illustrate the notion of the dependence of one quantity upon another (functional relationship)?

331. ***Theorem.*** *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*

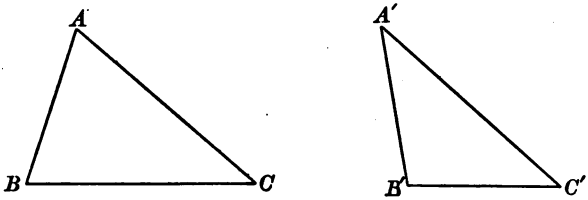


FIG. 339

Given the triangles *ABC* and *A′B′C′*, with *AB* equal to *A′B′*, *AC* equal to *A′C′*, and *BC* greater than *B′C′*.

To prove that $\angle A > \angle A'$.

Proof

STATEMENTS	REASONS
1. There are three possibilities, as follows: $\angle A < \angle A'$, $\angle A = \angle A'$, or $\angle A > \angle A'$.	1. Obvious.

Proof (*Continued*)

STATEMENTS	REASONS
2. If $\angle A < \angle A'$, then $BC < B'C'$.	2. If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.
3. But $BC > B'C'$, and hence $\angle A \not< \angle A'$.	3. Given.
4. If $\angle A = \angle A'$, then $\triangle ABC \cong \triangle A'B'C'$, and $BC = B'C'$.	4. If two triangles have two sides and the included angle of one triangle equal respectively to two sides and the included angle of the other, the triangles are congruent.
5. But $BC > B'C'$.	5. Given.
6. $\therefore \angle A > \angle A'$.	6. Because it is the only other possibility.

NOTE. The preceding theorem is applied in mechanics and physics work somewhat as follows: *As the side of a triangle is increased, the angle opposite it increases accordingly.*

EXERCISES

1. A simple form of crane consists of a movable beam BC (Fig. 340) hinged at B to a vertical mast AB. The beam is controlled by a wire or heavy rope attached to the beam at C and running over a pulley at A. If the rope is allowed to run free, the beam BC descends. How does this illustrate the truth of the theorems of Arts. 330 and 331?

2. Show how the crane in Fig. 340 illustrates in a mechanical way the dependence of one quantity upon another.

3. A diameter of a circle is greater than any other chord.

4. In the same circle or in congruent circles, if two minor arcs are unequal the greater arc has the greater chord.

SUGGESTION. Imagine the ⊙ O' (Fig. 341) to slide along the line $O'O$ until O' coincides with O. Then the circles will coincide. Why? Rotate the ⊙ O' about O until $\overset{\frown}{A'B'}$ falls within $\overset{\frown}{AB}$ as shown. Then prove that $AB > A'B$.

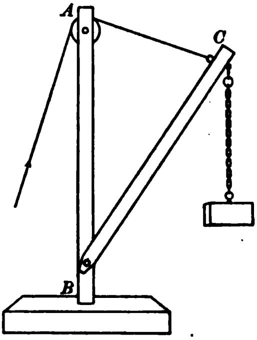


FIG. 340. CRANE

5. State and prove the converse of Ex. 4.

6. The shortest line which can be drawn to a circle from a point within the circle is the shorter segment of a diameter through the point.

7. The longest line which can be drawn to a circle from a point within the circle is the longer segment of the diameter through the point.

8. If P is a point within a $\triangle ABC$ such that $BP = BC$, prove that $AC > AP$.

9. The altitude from any vertex of a triangle is less than one half the sum of the two sides which meet at the vertex.

10. The sum of the three altitudes of a triangle is less than the sum of the three sides.

11. The perimeter of a triangle is less than twice the sum of the three medians.

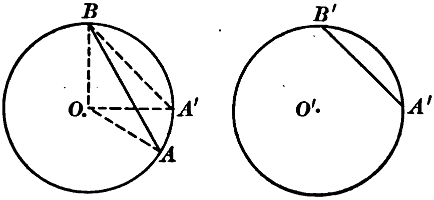


FIG. 341

12. The sum of either pair of opposite sides of a quadrilateral is less than the sum of its two diagonals.

13. If the perpendicular from any vertex of a triangle to the opposite side divides the side into two segments, how does each of these segments compare in length with the side of the triangle adjacent to it? Prove your answer.

14. Prove that the distance between the centers of two intersecting circles is less than the sum of the radii but is greater than the difference of the radii.

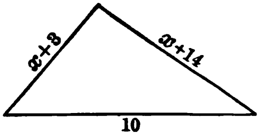


FIG. 342

15. In Fig. 342 determine between what limits x lies. What integral values could x represent?

332. Theorem. *In the same circle or in congruent circles unequal chords are unequally distant from the center, and the shorter chord lies at the greater distance from the center.*

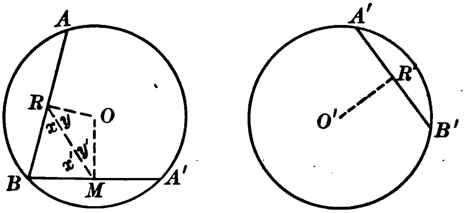


FIG. 343

Given the congruent circles O and O', with AB greater than $A'B'$.

To prove that $O'R' > OR$.

Proof

STATEMENTS	REASONS
1. Imagine the ⊙ O' to slide along the line $O'O$ until O' coincides with O.	1. Any geometric figure may be moved about from one position to another without changing its shape or size.
2. Rotate the ⊙ O' about O until B' falls on B and A' at A'.	2. A circle may be rotated about its center.
3. OR and OM are $\perp$ AB and BA' respectively.	3. "Distance from the center" means the perpendicular distance.
4. Draw RM.	4. One and only one straight line may be drawn between two points.

Proof (*Continued*)

STATEMENTS	REASONS
5. $\therefore AR = RB,$ and $BM = MA'.$	5. A line drawn from the center of a circle perpendicular to a chord bisects the chord.
6. But $AB > BA'.$	6. Given.
7. $\therefore RB > BM.$	7. Halves of unequals are unequal in the same order.
8. $\therefore \angle x' > \angle x.$	8. In a triangle in which two sides are unequal the greater angle lies opposite the greater side.
9. But $x + y = x' + y'.$	9. All right angles are equal.
10. $\therefore \angle y > \angle y'.$	10. If unequals are subtracted from equals, the results are unequal in the reverse order.
11. $\therefore OM > OR.$	11. In a triangle in which two angles are unequal the greater side lies opposite the greater angle.

EXERCISE

State and prove the converse of the preceding theorem.

333. *Theorem.* *A perpendicular is the shortest distance from a point to a plane.*

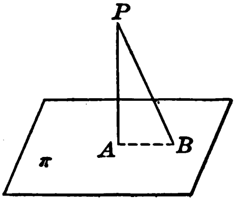


FIG. 344

Given the plane π and the line PA perpendicular to π at A.

***To prove that** PA is the shortest distance from P to π.*

Proof. The proof is left to the student.

NOTE. The length of the perpendicular from a point outside a plane to the plane is the distance from the point to the plane.

334. Theorem. *Oblique lines drawn from a point on a perpendicular to a plane and which meet the plane at equal distances from the foot of the perpendicular are equal; and of two oblique lines which meet the plane at unequal distances from the foot of the perpendicular the more remote is the greater.*

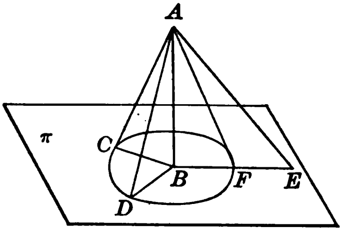


FIG. 345

I. Given AB perpendicular to the plane π, and AC, AD, and AF so drawn from A to the plane π that CB, DB, and BF are equal.

To prove that $AC = AD = AF$.

Proof. The proof is left to the student.

II. Given AB perpendicular to the plane π, and AD and AE so drawn from A to the plane π that BE is greater than BD.

To prove that $AE > AD$.

Proof. The proof is left to the student.

SUGGESTION. Imagine the $\triangle ADB$ to swing around AB as an axis until AD takes the position AF, which is coplanar with AE and AB. Then the theorem can be proved by plane geometry.

335. Theorem. *Equal oblique lines from a point on a perpendicular to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

Proof. The proof is left to the student.

336. ***Corollary 1.*** *The locus in space of a point equidistant from all points on a circle is a line through the center perpendicular to the plane of the circle.*

337. ***Corollary 2.*** *The locus in space of a point equidistant from the vertices of a triangle is a line through the center of the circumscribed circle perpendicular to the plane of the triangle.*

338. ***Corollary 3.*** *The locus in space of all points equidistant from two given points is the plane perpendicular to the midpoint of the line joining the given points.*

EXERCISES

1. The line perpendicular to the plane of a given triangle at the center of the inscribed circle is the locus of what points?

2. What theorem of plane geometry is analogous to Corollary 3, above? How do the proofs differ?

CHAPTER VII

RATIO; PROPORTION; VARIATION; SIMILAR POLYGONS

339. Ratio of two line segments. The student will ultimately have a clearer concept of a fraction if he will practice estimating one geometric magnitude as a fraction of another; as, for example, the lengths of two line segments, the areas of two rectangular garden plots, or the volumes of two stone tablets.

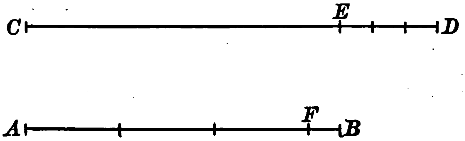


FIG. 346

The ratio of two line segments may be found by means of the compasses, as in the following example: In Fig. 346 *AB* and *CD* are two line segments, and we wish to find by means of the compasses what part *AB* is of *CD*.

We shall assume that these line segments have a common unit of measure, although we do not yet know what it is. To find the common unit of measure we lay off the shorter segment *AB* upon the longer segment *CD* as many times as possible (once in this case), leaving the remainder, *ED*. We then lay off *ED* on *AB* as many times as possible (three times in this case), leaving the remainder *FB*. We then lay off *FB* on the remainder *ED*. We see that it is contained exactly three times in *ED*. Hence the last remainder *FB* is the *common unit of measure*; that is, it is contained an integral number of times in both *AB* and *CD*. The ratio of the lines can then be found as in Exs. 1–3, below.

EXERCISES

1. Show that AB contains 10 units; that is, that $AB = 10\,FB$.

2. Show that CD contains 13 units.

3. Show that the ratio $\frac{AB}{CD}$ is equal to $\frac{10}{13}$.

4. Find by means of the compasses the common unit of measure of 72 and 16; of 68 and 14; of 51 and 12.

340. Magnitudes with no common unit of measure. By a process similar to that used in Art. 339 it can be shown that two magnitudes may not have a common unit of measure. A good example of two such magnitudes is that of the diagonal and a side of a square. For example, in Fig. 347 we know that

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2;$$

that is, $\overline{AB}^2 = 1^2 + 1^2,$

or $AB = \sqrt{2}.$

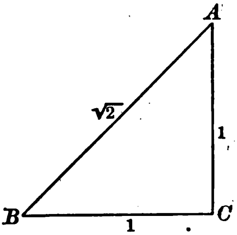


FIG. 347

That there is no way of finding a common unit of measure between AC and AB is readily shown by more advanced methods. The ratio of these two lines can, however, be found approximately and to any degree of accuracy desired.

For example, we may take the value of $\sqrt{2}$ as 1.4, as 1.41, as 1.414, as 1.4142, and so on. To make this clear, suppose that we try to find the common unit of measure of AC and AB. Let AC contain a certain unit of measure exactly 10 times. Then this same unit of measure is contained a little over 14 times in AB. Now let AC contain another smaller unit of measure exactly 100 times. Then this smaller unit of measure is contained a little over 141 times in AB. Again, let AC contain a still smaller unit of measure exactly 1000 times. Then this last unit of measure is contained a little over 1414 times in AB, and so on indefinitely. Thus, no matter how small we make the unit of measure which is contained an integral number of times in AC, it is never an exact divisor of AB. Hence no common unit of measure of AC and AB can ever be found by this method. We say, however, that for all practical purposes the ratio of AC to AB is equal to $\frac{1}{\sqrt{2}}$.

341. Commensurable and incommensurable magnitudes. We have seen in Art. 309 that if two magnitudes have a common unit of measure they are called *commensurable magnitudes*. If they do not have a common unit of measure, they are called *incommensurable magnitudes*.

EXERCISES

1. Find a common unit of measure between an angle of 35° and one of 120°.

2. In Fig. 348 show that $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$ are to each other as the central $\measuredangle s\ AOB$ and $A'O'B'$.

3. The arc of a circle is a function of the central angle.

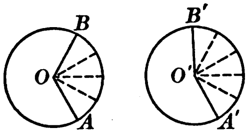


FIG. 348

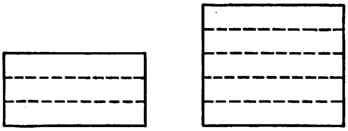


FIG. 349

4. The areas of the two rectangles in Fig. 349 are to each other as their altitudes.

342. Proportion. An equality of two or more ratios, as $\frac{a}{b}=\frac{c}{d}$, is called a *proportion*. The relation is a very simple example of the functional relationship between the four quantities involved. A series of equal ratios, like $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\cdots$, is called a *continual proportion.*

343. Means and extremes. In the proportion $\frac{a}{b}=\frac{c}{d}$ the first and last terms, a and d, are called the *extremes*, and the second and third terms, b and c, are called the *means*, of the proportion.

EXERCISES

1. If four quantities are in proportion, the product of the means equals the product of the extremes; that is, if $\frac{a}{b}=\frac{c}{d}$, prove that $bc = ad$.

2. Using the theorem of Ex. 1, solve the following equations:

(a) $\frac{x}{56}=\frac{8}{14}$.

(b) $\frac{8}{x}=\frac{6}{9}$.

(c) $\frac{3x+2}{14}=\frac{4}{5}$.

(d) $\frac{x-12}{1-3x}=\frac{6}{4}$.

(e) $\frac{y-4}{y-3}=\frac{2y+10}{2y+6}$.

(f) $\frac{3y+2}{y-4}=\frac{3y-7}{y+5}$.

3. If $bc = ad$, then

(a) $\frac{a}{b}=\frac{c}{d}$. (b) $\frac{a}{c}=\frac{b}{d}$. (c) $\frac{d}{b}=\frac{c}{a}$. (d) $\frac{d}{c}=\frac{b}{a}$.

4. Show that the results of Ex. 3 prove the following theorem: *If the product of two numbers is equal to the product of two other numbers, either pair may be made the means and the other pair the extremes of a proportion.*

NOTE. The results of Ex. 3 above are often stated in theorem form as follows:

If four quantities are in proportion, they are in proportion

1. *By alternating (interchanging) the means.*
2. *By alternating the extremes.*
3. *By alternating both the means and extremes* (inversion).

5. If three terms of one proportion are equal respectively to three corresponding terms of another proportion, then the fourth terms are also equal.

SUGGESTION. In each case solve the equation for the fourth term and see if the results are equal to the same thing.

6. Form proportions from the following:

(a) $mn = xy$.
(b) $ab = c^2 - d^2$.

SUGGESTION. Factor the right member.

(c) $a^2 - b^2 = 2m^2 - 2n^2$.
(d) $8a^2 - amn = am + an - ar$.
(e) $x^4 - y^4 = y^2 - z^2$.
(f) $ax + ay + az = mn + mr + ms$.
(g) $x^2 + 5x + 6 = y^2 - y - 6$.

344. Mean proportional. In the proportion $\frac{a}{b} = \frac{b}{c}$, b is called the *mean proportional* between a and c.

EXERCISES

1. In the proportion $\frac{a}{b} = \frac{b}{c}$ show that $b = \pm\sqrt{ac}$ (read "b equals plus or minus the square root of ac").

2. Find a mean proportional between 4 and 400; between 20 and 20; between 42 and 24; between x^2 and y^2; between $4m^3n$ and $9mn$.

345. Antecedent; consequent. In the proportion $\frac{a}{b} = \frac{c}{d}$ the numerators a and c are called *antecedents*, and the denominators b and d are called *consequents*.

EXERCISE

Form the ratio of the sum of the antecedents to the sum of the consequents in the proportion $\frac{2}{3} = \frac{8}{12}$. How does this ratio compare with the ratio in the given proportion?

346. *Theorem.* *If any number of quantities are in proportion, then the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\cdots.$

To prove that $\frac{a+c+e+g+\cdots}{b+d+f+h+\cdots}=\frac{a}{b}=\frac{c}{d}=\cdots.$

Proof

STATEMENTS	REASONS
1. $\frac{a}{b}=\frac{a}{b}.$	1. Identical.
2. $\frac{c}{d}=\frac{a}{b}.$	2. Given.
3. $\frac{e}{f}=\frac{a}{b}.$	3. Given.
4. $\frac{g}{h}=\frac{a}{b}$, and so on.	4. Given.
5. ∴ from 1, $ab = ab.$	5. If four quantities are in proportion, the product of the means equals the product of the extremes.
6. In like manner, from 2, $cb = ad.$	6. For the same reason.
7. And from 3, $eb = af.$	7. For the same reason.
8. Also from 4, $gb = ah.$	8. For the same reason.
9. Adding the results of 5, 6, 7, and 8, we have $ab + cb + eb + gb + \cdots$ $= ab + ad + af + ah + \cdots.$	9. Equals added to equals give equals.
10. Then $(a + c + e + g + \cdots)b$ $= a(b + d + f + h + \cdots).$	10. By factoring out the common monomial factor in each member of the equation.

Proof (*Continued*)

STATEMENTS	REASONS
11. $\therefore \dfrac{a+c+e+g+\cdots}{b+d+f+h+\cdots} = \dfrac{a}{b} = \cdots.$	11. If the product of two quantities equals the product of two other quantities, either pair may be made the means and the other pair the extremes of a proportion.

347. Theorem. *If four quantities are in proportion, they are in proportion by addition.*

Given $\dfrac{a}{b}=\dfrac{c}{d}.$

To prove that $\dfrac{a+b}{b}=\dfrac{c+d}{d}.$

Analysis

STATEMENTS	REASONS
1. Assume that $\dfrac{a+b}{b}=\dfrac{c+d}{d}.$	1. Obviously we can make this assumption.
2. Then $(a+b)d=b(c+d).$	2. By multiplying both sides of the equation by *bd*.
3. From 2, $ad+bd=bc+bd.$	3. By performing the indicated multiplications.
4. $\therefore ad=bc.$	4. By subtracting *bd* from each member of the equation.
5. $\therefore \dfrac{a}{b}=\dfrac{c}{d}.$	5. If the product of two factors is equal to the product of two other factors, either pair may be made the means and the other pair the extremes of a proportion.

NOTE. In the proof the preceding steps are merely retraced, as shown at the top of the following page. The student should study these steps very carefully in order that he may learn to use the method therein illustrated in other problems.

Proof

STATEMENTS	REASONS
1. $\frac{a}{b}=\frac{c}{d}$.	1. Given.
2. $ad = bc$.	2. If four quantities are in proportion, the product of the extremes is equal to the product of the means.
3. $ad + bd = bc + bd$.	3. By adding bd to both members of the equation.
4. Then $d(a + b) = b(c + d)$.	4. By factoring.
5. $\therefore \frac{a+b}{b}=\frac{c+d}{d}$.	5. If the product of two quantities equals the product of two other quantities, either pair may be made the means and the other pair the extremes of a proportion.

348. Method of analysis. The method of proving the theorem of Art. 347 is called the *method of analysis*, and is very useful in proving many algebraic equalities. The first part of the method is the analysis, in which we assume the conclusion to be true and from which, by the fundamental processes of algebra and by reasoning, we reach some known fact. The second part is the proof, in which we practically reverse the steps of the analysis and obtain the conclusion. Inasmuch as this reversal of steps is not always possible, the student should never fail to complete the proof step by step.

EXERCISES

1. Prove by analysis that if four quantities are in proportion, they are in proportion by subtraction; that is, if

$$\frac{a}{b}=\frac{c}{d},$$

then

$$\frac{a-b}{b}=\frac{c-d}{d}.$$

2. Prove by analysis that if four quantities are in proportion, they are in proportion by addition and subtraction; that is, if

$$\frac{a}{b}=\frac{c}{d},$$

then

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

349. Proportion and variation. Many problems in physics, chemistry, general science, mathematics, and astronomy may be solved by treating them either as problems in proportion or as problems in variation. In fact, both proportion and variation exhibit functional relationships so well that the student may usually state his problem in either way if he clearly understands the kind of dependence involved.

350. Direct variation. When two numbers vary so that one depends upon the other for its value and the ratio of any value of one to the corresponding value of the other remains constant, then one number is said to *vary directly as the other*, or to be *directly proportional* to the other. Thus, the number y is said to vary directly as x if the ratio $\frac{y}{x}$ remains constant as x and y vary. The equation $\frac{y}{x}=k$ expresses algebraically the statement that "y varies directly as x," k being called the *constant of variation* and x and y the *variables*.

EXERCISES

1. In the equation $\frac{y}{x}=k$ show that y is a function of x. Draw the graph of $\frac{y}{x}=10$.

2. Give five concrete examples which illustrate direct variation.

3. If y varies directly as x and $x=6$ when $y=2$, find y when $x=8$.

4. The time of oscillation of a pendulum varies directly as the square root of its length. What is the length of a pendulum which makes an oscillation in 4 sec. if a pendulum 156.8 in. long makes an oscillation in 2 sec.?

351. Dependent and independent variables. In the equation $\frac{y}{x} = k$, y is said to depend upon x for its value and hence is called the *dependent variable.* Since x is said to be independent of y, it is called the *independent variable.* The dependent variable is usually placed on the left side of an equation.

352. Inverse variation. When two numbers vary so that the product of any value of one number and the corresponding value of the other is a constant, then the one number is said to *vary inversely* as the other, or to be *inversely proportional* to the other. Thus, the number x is said to vary inversely as y if the product xy remains constant as x and y vary. The equation $xy = k$ expresses algebraically the statement "x varies inversely as y."

EXERCISES

1. In the equation $xy = k$ show that x is a function of y. Draw the graph of $xy = 5$.

2. Give two examples which illustrate inverse variation.

3. If x varies inversely as y and $x = 5$ when $y = 2$, what is the value of x when $y = 4$?

4. In a compressed-air pump the pressure of the air varies inversely as its volume. If the pressure is 25 lb. per square inch when the volume is 140 cu. in., what is the pressure when the volume is 125 cu. in.?

5. The number of men engaged on a piece of work varies inversely as the time required to complete it. If 10 men can do a piece of work in 33 da., how long will it take 12 men to do the same piece of work? Solve first as a problem in variation and then check your result by solving as a problem in proportion.

6. If x varies inversely as y and $x = \frac{3}{4}$ when $y = \frac{2}{3}$, what is the value of x when y is $\frac{5}{6}$?

7. Fig. 350 represents one type of loaded bar, or lever. The bar AB is supported at F, the fulcrum, and a weight W is suspended at B. If the proper amount of force (pressure) is exerted downward at A, the bar can be made to balance. The law of levers is expressed by the equation $\frac{P}{W} = \frac{y}{x}$, which may be translated as follows: The force and weight are inversely proportional to their distances from the fulcrum. If $x = 12$ ft., $y = 10$ ft., and a weight of 50 lb. is suspended at B, what force must be exerted to lift the weight?

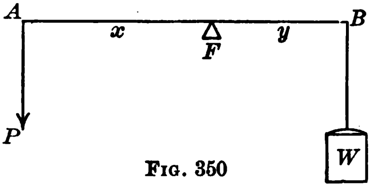


FIG. 350

8. In Ex. 7, if $x = 15$ ft., $y = 12$ ft., and $P = 250$ lb., what weight can be lifted at B?

9. In Ex. 7, if $P = 250$ lb., $W = 75$ lb., and $x = 14$ ft., what is the length of y?

10. Fig. 351 illustrates the law of physics known as the *law of inverse squares*, which states that the intensity of light varies inversely as the square of the distance from the source. Stated symbolically, the law is $Id^2 = k$, where I represents the intensity of the light, d the distance of the screen from the source of light, and k the constant of variation. Explain the figure from the formula.

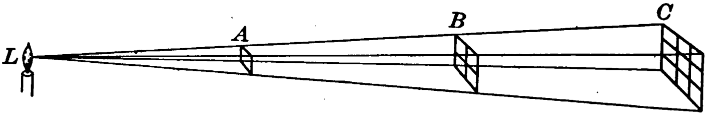


FIG. 351

11. What happens to the intensity (I) in Ex. 10 if d is doubled? if d is trebled?

12. How does the formula $Id^2 = k$ illustrate functional relationship?

353. *Theorem.* *A line parallel to one side of a triangle divides the other two sides proportionally.*

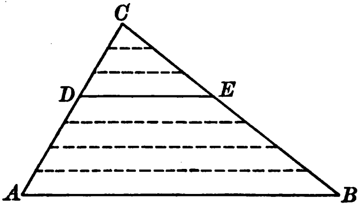


FIG. 352

Given the triangle *ABC*, with *DE* parallel to *AB*.

To prove that $\frac{CD}{DA} = \frac{CE}{EB}$.

Proof

STATEMENTS	REASONS
1. Suppose that CD and DA are commensurable and that x is the common unit of measure.	1. We have agreed to consider only commensurable cases.
2. Let x be contained in CD m times and in DA n times.	2. Since the segments are assumed to be commensurable.
3. Then $\frac{CD}{DA} = \frac{mx}{nx} = \frac{m}{n}$.	3. By definition of the ratio of two line segments.
4. Through the points of division of CD and DA draw lines $\parallel AB$.	4. Through a point outside a given line one and only one line can be drawn parallel to the line.
5. These parallel lines would divide CE into m equal parts and EB into n equal parts.	5. If two or more parallel lines cut off equal segments on one transversal, they cut off equal segments on every transversal.
6. On CE and EB call the length of each segment cut off y. Then $\frac{CE}{EB} = \frac{my}{ny} = \frac{m}{n}$.	6. By definition of the ratio of two line segments.
7. $\therefore \frac{CD}{DA} = \frac{CE}{EB}$.	7. Things equal to the same thing are equal to each other.

354. Corollary. *If a line is parallel to the side AB of the* $\triangle ABC$ *and cuts AC at D and BC at E, then*

$$\frac{AC}{AD}=\frac{BC}{BE}, \quad \textit{and} \quad \frac{AC}{DC}=\frac{BC}{EC}.$$

SUGGESTION. Apply addition to the conclusion of Art. 353.

EXERCISES

1. State the above corollary without letters or a proportion.

2. Using the figure in Art. 353, prove that

(a) $\frac{AD}{AC}=\frac{BE}{BC}$. (b) $\frac{DC}{AC}=\frac{EC}{BC}$. (c) $\frac{DA}{CD}=\frac{EB}{CE}$.

3. In Fig. 352, if $AD=9$ in., $CD=7$ in., and $EC=5$ in., find BE.

4. In Fig. 352, if $AC=16$ in., $BC=20$ in., and $BE=8$ in., find AD.

5. If two parallel lines cut two intersecting transversals, the segments of one transversal are proportional to the corresponding segments of the other.

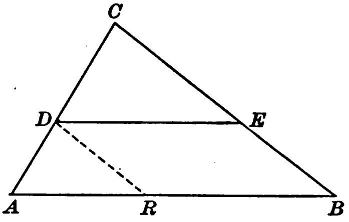


FIG. 353

6. If any number of parallel lines cut two transversals, the segments of one transversal are proportional to the corresponding segments of the other.

7. If two given lines are cut by two parallel lines, the segments of the parallel lines are proportional to the corresponding segments of the given lines.

SUGGESTION. Using Fig. 353, draw $DR \parallel EB$ and prove that $\frac{CD}{CA}=\frac{DE}{AB}$.

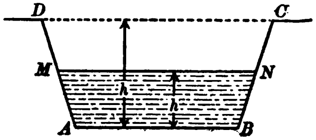


FIG. 354

8. Fig. 354 represents a trapezoidal cross section of a canal, the water level being shown by the line MN. If the lines MN and AB are horizontal, show that the dry segments DM and CN are proportional to the wet segments MA and NB.

355. Theorem. *If three or more parallel planes are cut by two transversals, the corresponding segments of the transversals are proportional.*

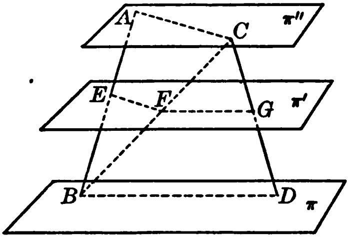


FIG. 355

Given the parallel planes π, π', and π'' cut by the transversals AB and CD.

To prove that $\dfrac{AE}{EB} = \dfrac{CG}{GD}$.

Proof

STATEMENTS	REASONS
1. Draw BC intersecting the plane π' in F.	1. One and only one straight line can be drawn between two points.
2. Imagine a plane passed through AB and BC, cutting the plane π' in EF; also one through BC and CD, cutting the plane π' in FG.	2. Two intersecting lines determine a plane, and two planes intersect in a straight line.
3. Then $EF \parallel AC$, and $FG \parallel BD$.	3. If two parallel planes are cut by a third plane, the lines of intersection are parallel.
4. $\therefore \dfrac{AE}{EB} = \dfrac{CF}{FB}$, and $\dfrac{CG}{GD} = \dfrac{CF}{FB}$.	4. If two parallel lines are cut by two intersecting transversals, the corresponding segments are proportional.
5. $\therefore \dfrac{AE}{EB} = \dfrac{CG}{GD}$.	5. Things equal to the same thing are equal to each other.

356. Corollary. *If two parallel planes cut any number of concurrent lines, the lines are divided proportionally by their intersection point and the two planes.*

EXERCISES

1. In Fig. 355 AB cuts the three parallel planes so that $AE = 12$, $EB = 14$, and $CD = 22$. Find CG and GD.

2. In Fig. 355, if $AB = 10$, $CG = 4$, and $CD = 9$, find AE and EB.

357. Theorem. *If a straight line divides two sides of a triangle proportionally, the line is parallel to the third side.*

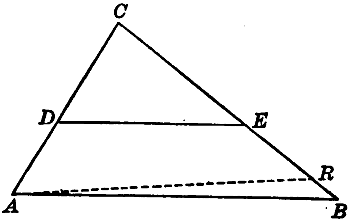


FIG. 356

Given the triangle *ABC*, with *DE* drawn such that

$$\frac{CD}{DA} = \frac{CE}{EB}.$$

To prove that $DE \parallel AB$.

Proof

STATEMENTS	REASONS
1. Suppose that AB is not ∥ DE and that some other line, as AR, is ∥ DE.	1. It is clear that through A there must be some line parallel to DE.
2. Then $\frac{CD}{DA} = \frac{CE}{ER}$.	2. A line parallel to the base of a triangle divides the sides proportionally.
3. But $\frac{CD}{DA} = \frac{CE}{EB}$.	3. Given.

Proof (*Continued*)

STATEMENTS	REASONS
4. $\frac{CE}{ER}=\frac{CE}{EB}$.	4. Things equal to the same thing are equal to each other.
5. $\therefore CE \cdot ER = EB \cdot CE$.	5. If four quantities are in proportion, the product of the means equals the product of the extremes.
6. $\therefore ER = EB$.	6. Dividing both members of the equation by CE.
7. But this is absurd, for $ER < EB$.	7. Obvious.
8. $\therefore$ the assumption that AB is not $\parallel$ DE is false, and $DE \parallel AB$.	8. By eliminating the other possibility.

EXERCISES

1. The line connecting the midpoints of two sides of a triangle is parallel to the third side.

2. Three or more lines which cut off proportional segments on two given intersecting lines are parallel.

3. The median of a trapezoid is parallel to the bases.

4. If the midpoints of a skew quadrilateral are joined, the figure thus formed is a parallelogram.

358. Internal and external division of a line segment. If on a line segment AB (Fig. 357) a point P is taken between A and B, the line AB is said to be *divided internally* by the point P. In reading the two segments we read from the end A to the point of division P (the segment AP) and from P to the other end B (the segment PB). If the point P' is taken to the right of B, as shown in Fig. 357, then the line segment AB is said to

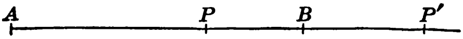


FIG. 357

be *divided externally* by the point P'. The segments are read as before from one end of the line A to the point of division P' (the segment AP') and from P' to the other end B (the segment $P'B$). In either case, if we consider the direction of AB as positive and the direction of BA as negative, then $AP + PB = AB$.

359. Theorem. *The bisector of an angle of a triangle divides the opposite side internally into segments proportional to the adjacent sides.*

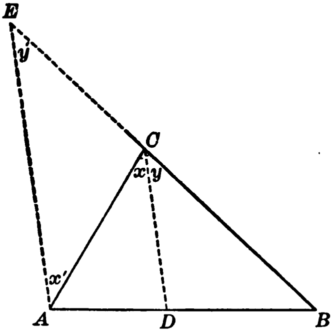


FIG. 358

Given the triangle *ABC*, with angle *x* equal to angle *y*.

To prove that $\frac{AD}{DB} = \frac{AC}{CB}$.

Proof

STATEMENTS	REASONS
1. Draw $AE \parallel CD$ and produce BC to E.	1. Through a point outside a given line one and only one line can be drawn parallel to the line.
2. Then $\frac{AD}{DB} = \frac{EC}{CB}$.	2. If two parallel lines cut two intersecting transversals, the corresponding segments are proportional.

Proof (*Continued*)

STATEMENTS	REASONS
3. $\angle y = \angle y'$.	3. If two parallel lines are cut by a transversal, the corresponding angles are equal.
4. $\angle x = \angle x'$.	4. If two parallel lines are cut by a transversal, the alternate-interior angles are equal.
5. But $\angle x = \angle y$.	5. Given.
6. $\therefore \angle x' = \angle y'$.	6. Things equal to equal things are equal to each other.
7. $\therefore EC = AC$.	7. If the base angles of a triangle are equal, the triangle is isosceles.
8. $\therefore \frac{AD}{DB} = \frac{AC}{CB}$.	8. By substitution.

EXERCISES

1. State and prove the converse of the preceding theorem.

2. In Fig. 358, if $AB = 10$, $AC = 12$, and $CB = 14$, find AD and DB.

3. In Fig. 358, if $AC = 8$, $CB = 10$, and $DB = 7$, find AB and AD.

4. The sides of a given triangle are 10 in., 20 in., and 14 in. respectively. Find the segments of the side of length 14 in. made by the bisector of the angle opposite it.

5. The sides of a triangle are 7 in., 8 in., and 9 in. respectively. Find the segments of each side made by the bisector of the opposite angle.

6. Prove that a straight line which is drawn through a vertex of a triangle so as to divide the opposite side internally into segments proportional to the adjacent sides bisects the angle at that vertex.

SUGGESTION. Using a figure like Fig. 358, produce BC to E until $CE = CA$. Then prove that $\angle x = \angle y$.

360. *Theorem.* *The bisector of the exterior angle of a triangle divides the opposite side externally into segments proportional to the other two sides.*

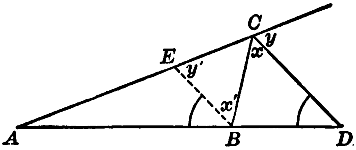


Fig. 359

Given the triangle *ABC*, with angle *x* equal to angle *y*.

To prove that $\frac{AD}{DB} = \frac{AC}{CB}$.

Proof

Statements	Reasons
1. Draw $EB \parallel CD$.	1. Through a point outside a given line one and only one line can be drawn parallel to the line.
2. Then $\frac{AD}{DB} = \frac{AC}{CE}$.	2. If two parallel lines are cut by two intersecting transversals, the corresponding segments are proportional.
3. $\angle y = \angle y'$.	3. If two parallel lines are cut by a transversal, the corresponding angles are equal.
4. $\angle x = \angle x'$.	4. If two parallel lines are cut by a transversal, the alternate interior angles are equal.
5. But $\angle x = \angle y$.	5. Given.
6. $\therefore \angle x' = \angle y'$.	6. Things equal to equal things are equal to each other.
7. $\therefore EC = CB$.	7. If the base angles of a triangle are equal, the triangle is isosceles.
8. $\therefore \frac{AD}{DB} = \frac{AC}{CB}$.	8. By substitution.

EXERCISES

1. In Fig. 359, if $AB = 7$, $AC = 8$, and $BC = 9$, find AD and BD.

2. By means of the theorem of Art. 359 prove that the bisector of the vertex angle of an isosceles triangle bisects the base.

3. State and prove the converse of the theorem of Art. 360.

4. The sides of a triangle are 10 in., 12 in., and 14 in. respectively. Find the segments into which the 14-inch side is divided by the bisector of the exterior angle at the opposite vertex.

5. In a certain triangle one side is 8 in. long. The angle opposite this side is bisected, and the bisector divides this side into two segments such that if 4 in. be added to the shorter, it is equal to one of the sides of the triangle; and if 3 in. be added to twice the shorter segment, it is equal to the other side. Find the length of the segments into which the 8-inch side is divided.

361. Harmonic division of a line segment. If a line segment is divided internally and externally in the same ratio, the line segment is said to be *divided harmonically.*

EXERCISE

Prove that the bisector of the interior angle of a triangle and the bisector of the exterior angle at the same vertex divide the opposite side harmonically.

362. Dividing a line segment in a given ratio. To divide a given line segment AB in the ratio $\frac{m}{n}$ means to find a point P on AB such that it divides AB internally in the ratio $\frac{m}{n}$, or to find a point P' on AB extended such that it divides AB externally in the ratio $\frac{m}{n}$.

363. *Construction problem.* *To divide a line segment internally in the ratio* $\frac{m}{n}$.

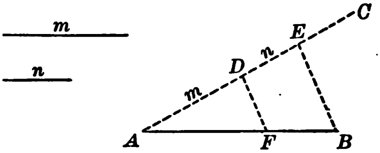


FIG. 360

Given the line segment *AB*.

Required *to divide AB internally in the ratio* $\frac{m}{n}$.

Construction. Draw a line AC, making any convenient angle with AB, and lay off $AD = m$ and $DE = n$. Connect B and E, and draw $DF \parallel EB$.

Then F divides AB internally in the ratio $\frac{m}{n}$.

Proof. The proof is left to the student.

364. *Construction problem.* *To divide a line segment externally in the ratio* $\frac{m}{n}$.

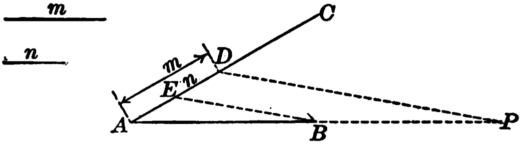


FIG. 361

Given the line segment *AB*.

Required *to divide AB externally in the ratio* $\frac{m}{n}$.

Construction. Draw AC at any convenient angle with AB, and lay off $AD = m$ and $DE = n$. Connect B and E, and draw $DP \parallel EB$.

Then $\frac{AP}{PB} = \frac{m}{n}$.

Proof. The proof is left to the student.

EXERCISE

Show how a line segment may be divided internally and externally in the same ratio by using the theorems of Arts. 359 and 360.

365. *Construction problem.* *To divide a given line segment into segments proportional to several given segments.*

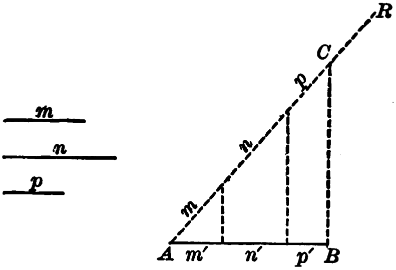


Fig. 362

Given the line segment *AB* and the segments *m*, *n*, and *p*.

Required *to divide AB into segments proportional to m, n, and p.*

Construction. Draw AR making any convenient angle with AB. On AR lay off m, n, and p in succession, as shown in Fig. 362. Join B to C, the last point of division. Draw lines ∥ BC as shown.

Then $\frac{m}{m'} = \frac{n}{n'} = \frac{p}{p'}$.

Proof. The proof is left to the student.

EXERCISES

1. Draw a line segment 12 units long and divide it internally and externally in the ratio of 2 to 3.

2. Prove that if a line segment AB is divided harmonically by two points C and D, the line segment CD is divided harmonically by the points A and B.

3. Show how the theorem of Art. 365 can be used to bisect a given line segment.

366. ***Construction problem.*** *To construct a fourth proportional to three given line segments.*

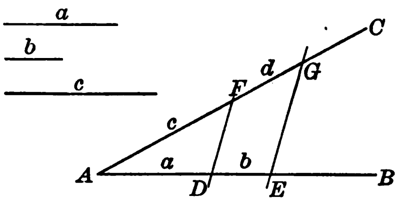


FIG. 363

Given the line segments *a*, *b*, and *c*.

Required *to construct a fourth proportional to a, b, and c.*

Construction. Draw any convenient $\angle BAC$. Then on AB lay off $AD = a$ and $DE = b$. On AC lay off $AF = c$. Draw DF, and then draw $EG \parallel DF$ and cutting AC at G.

Then FG is the required fourth proportional.

Proof. The proof is left to the student.

EXERCISES

1. To construct a rectangle equal to a given parallelogram and having one side equal to a given line segment.

SUGGESTION. Let x (Fig. 364) represent the side of the required rectangle. Then, since it is required that the area of the rectangle equal the area of the parallelogram, ah must equal bx.

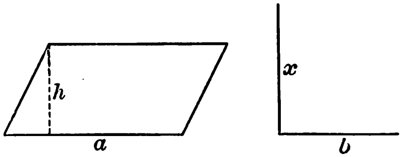


FIG. 364

2. To construct a rectangle three times as large as a given parallelogram and having one side equal to a given line segment.

3. Show how a given line segment may be divided in any given ratio by laying it off on squared paper.

367. *Construction problem.* *To divide a triangle in a given ratio by a line parallel to a given side.*

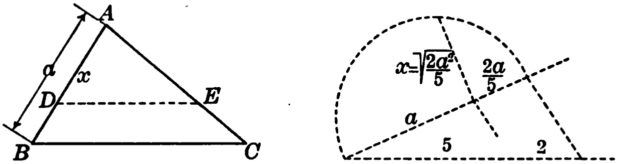


FIG. 365

Given the triangle *ABC* and the ratio $\frac{2}{3}$.

Required *to divide the* $\triangle ABC$ *in the ratio* $\frac{2}{3}$ *by a line parallel to* BC.

Analysis. Imagine the construction done as required by using the line DE (Fig. 365).

1. Then $\frac{\triangle ADE}{DBCE} = \frac{2}{3}$, by construction.

2. From 1, $\frac{\triangle ADE}{\triangle ABC} = \frac{2}{5}$, by addition.

3. But $\frac{\triangle ADE}{\triangle ABC} = \frac{x^2}{a^2}$, because the areas of two similar triangles are to each other as the squares of any two homologous sides.

4. $\therefore \frac{x^2}{a^2} = \frac{2}{5}$, because things equal to the same thing are equal to each other.

5. Then $x^2 = \frac{2\,a^2}{5}$, or $x^2 = \frac{2\,a.a}{5} = a\,\frac{2\,a}{5}$, which means that we must first find $\frac{2\,a}{5}$, then a, and then x is a mean proportional between a and $\frac{2\,a}{5}$.

Construction. See Ex. 1 on page 286.

NOTE. The theorem quoted in step 3 above is not proved until Art. 493, but we shall take it for granted here. This is in agreement with our practice of occasionally taking for granted a few fundamental principles without proof, the more formal proof to be given at a later stage. Much of the work that is done in mathematics is done in this way.

EXERCISES

1. To find $\frac{2a}{5}$ let $y = \frac{2a}{5}$, and then y is simply a fourth proportional to 5, a, and 2. The student should now complete the preceding construction problem and prove it.

2. How far from the vertex of a triangle whose altitude is 12 must a line be drawn parallel to the base to divide the triangle into two equal parts?

3. Divide a triangle into three equal parts by lines drawn parallel to the base.

368. Similar figures. The two leaves shown in Fig. 366 have the same shape but not the same size. The same is true of the two fans shown in Fig. 367.

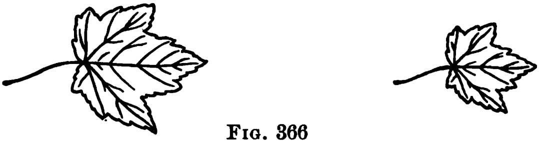


FIG. 366

It is evident that the two objects of each pair have great similarity of appearance. Figures possessing such properties are called *similar figures*. It is readily seen that the two figures in Fig. 368 are similar.

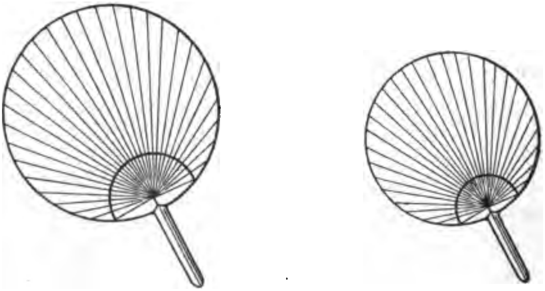


FIG. 367

369. Similar polygons. Polygons which have the same shape are called *similar polygons.* In order that two polygons shall have the same shape, it is necessary and sufficient that they have their homologous angles equal and their homologous sides proportional. Thus, we may define similar polygons as *polygons which are mutually equiangular and whose homologous sides are in proportion.* The polygons $ABCDE$ and $A'B'C'D'E'$ in Fig. 369 are similar polygons.

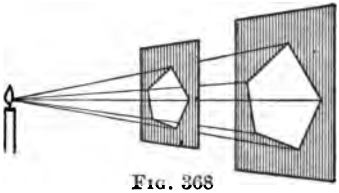


FIG. 368

370. Ratio of similitude. The ratio of two homologous sides of two similar polygons is called the *ratio of similitude* of the polygons.

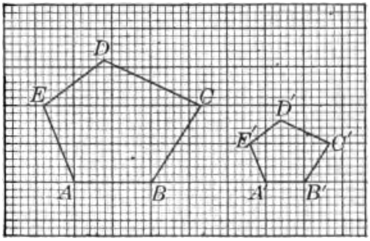


FIG. 369

371. Uses of similarity. The student has perhaps observed that throughout his course in mathematics he has either enlarged or reduced many of the figures in the text in order to make his work in connection with the figures clearer or easier. In nearly all books the drawings which represent real objects are either enlarged or reduced from the actual size. These enlargements or reductions from the actual size are made by increasing or reducing all the dimensions of the drawing in the same ratio, so that all the dimensions of the drawing are proportional to the homologous dimensions of the object.

The plans used in construction work, shop drawings, a surveyor's map of a triangular field, blueprints, and photographs are all examples of the uses of similarity. Thus, you may be able to tell by looking at a photograph that a man has large ears, although in the photograph neither of his ears is a centimeter in length. You can tell by looking at a plan of a house whether the windows are large or small, because the relative size is shown just as well in the plan, since all parts are reduced in the same scale; that is, *the ratios of all homologous parts of plan and object are equal.* Two similar triangles may be thought of as copies of a given triangle, one enlarged and one reduced in the same proportion; or both may be regarded as copies of the given triangle to different scales.

EXERCISES

1. Find the ratio of the homologous sides of the similar polygons in Fig. 369.

2. Are two rectangles necessarily similar? two squares? two equilateral triangles? Give the reasons for your answer.

3. The sides of one triangle are 1, 2, and 3, and the longest side of a similar triangle is 8. What are the lengths of the other sides of the second triangle?

4. If the dimensions of a certain rectangle are 8 and 10, what are the dimensions of a similar rectangle that has a perimeter of 48?

5. If a map is drawn to the scale $\frac{1}{10000}$, what lengths represent the dimensions of a rectangular county 20 mi. long and 10 mi. wide?

372. Similarity of triangles. From the definition of similar polygons given in Art. 369 the student will see that the necessary and sufficient conditions for the similarity of two triangles are two in number, as follows:

1. The triangles are mutually equiangular.
2. The homologous sides of the triangles are proportional.

However, it is possible to show that if some of the conditions are given, the others necessarily follow. Thus, we shall show that if we wish to prove that two triangles are similar, it is sufficient to show that

1. Two angles of one are equal to two homologous angles of the other.
2. Two sides of one are proportional to two homologous sides of the other, and that the angles included between these pairs of sides are equal.
3. The three homologous sides are proportional.

373. *Theorem.* *If two angles of one triangle are equal respectively to two angles of another triangle, the triangles are similar.*

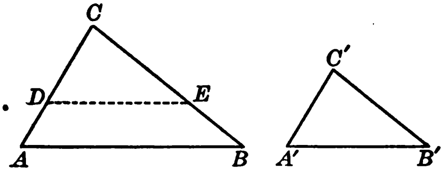


FIG. 370

Given the triangles *ABC* and *A'B'C'*, with angle *A* equal to angle *A'* and angle *C* equal to angle *C'*.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Imagine the $\triangle A'B'C'$ placed on the $\triangle ABC$ so that $\angle C'$ coincides with its equal $\angle C$, and so that $CD = C'A'$ and $CE = C'B'$.	1. Any geometric figure may be moved about in space from one position to another without changing its size or shape.
2. Draw DE.	2. One and only one straight line can be drawn between two points.

Proof (*Continued*)

STATEMENTS	REASONS
3. Then $\triangle CDE \cong \triangle A'C'B'$.	3. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
4. $\therefore \angle CDE = \angle A'$.	4. Homologous parts of congruent triangles are equal.
5. But $\angle A' = \angle A$.	5. Given.
6. $\therefore DE \parallel AB$.	6. If two lines are cut by a transversal which makes the corresponding angles equal, the lines are parallel.
7. $\therefore \frac{CD}{CA} = \frac{CE}{CB}$.	7. If two parallel lines are cut by two transversals, the corresponding segments are proportional.
8. $\therefore \frac{C'A'}{CA} = \frac{C'B'}{CB}$.	8. By substituting $C'A'$ and $C'B'$ for CD and CE respectively.
9. In like manner, $\frac{C'B'}{CB} = \frac{B'A'}{BA}$.	9. For similar reasons.
10. $\therefore \frac{A'C'}{AC} = \frac{B'C'}{BC} = \frac{B'A'}{BA}$.	10. Things equal to the same thing are equal to each other.
11. $\therefore \triangle A'B'C' \sim \triangle ABC$.	11. By definition of similar triangles.

NOTE. The symbol $\sim$ is read "is similar to."

EXERCISES

1. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

2. If a line is drawn parallel to any side of a triangle, it forms with the other two sides a triangle which is similar to the given triangle.

3. If two triangles have the sides of one respectively parallel or perpendicular to the sides of the other, the triangles are similar (see Fig. 371).

4. Two isosceles triangles are similar if the vertex angle of one is equal to the vertex angle of the other.

5. If each of two triangles are similar to a third triangle, they are similar to each other.

6. If the midpoints of the three sides of a given triangle are joined, the triangle thus formed is similar to the given triangle.

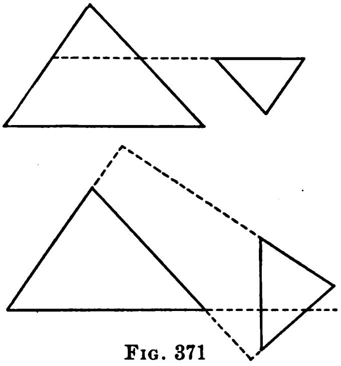


FIG. 371

374. Similar triangles and proportional lines. One of the commonest ways of proving that four line segments are proportional is to show that they are homologous sides of similar triangles. In order to make the proper proof we must observe the following directions:

1. Choose the two triangles so that each contains two of the given lines.
2. Prove that the triangles are similar.
3. Remember that homologous sides of similar triangles are the sides which lie opposite equal angles.
4. Write a proportion containing the proper sides.
5. If necessary, apply the principles of alternation or inversion to get the correct proportion.

EXERCISES

1. If in the $\triangle ABC$ the altitudes AD and CE are drawn, prove that $\frac{AB}{BC} = \frac{BD}{BE}$.

2. In two similar triangles the homologous bisectors of the angles have the same ratio as any two homologous sides.

3. The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.

4. The homologous medians of two similar triangles have the same ratio as any two homologous sides.

5. The diagonals of a trapezoid divide each other proportionally.

6. In Fig. 372 AE and BD are altitudes on the sides BC and AC respectively. Prove that $\frac{AE}{BD} = \frac{AC}{BC}$.

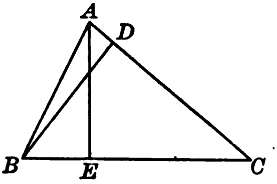


FIG. 372

7. The product of the two legs of a right triangle is equal to the product of the hypotenuse and the perpendicular from the vertex of the right angle upon the hypotenuse.

375. *Theorem.* *If two chords intersect within a circle, the product of the segments of one chord is equal to the product of the segments of the other.*

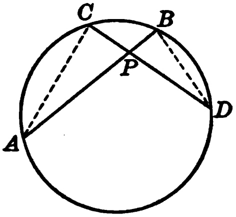


FIG. 373

SUGGESTION. Draw AC and BD.

EXERCISES

1. In Fig. 373, if $AP = 10$, $PB = 2$, and $PD = 7$, find PC.

2. In Fig. 373, if $CP = 4$, $PD = 6$, and $AB = 14$, find AP and PB.

376. Theorem. *If from a point outside a circle two secants terminating on the circle are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*

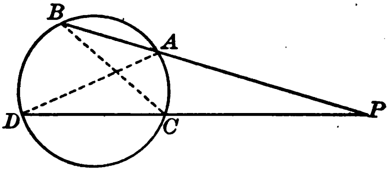


FIG. 374

SUGGESTION. Draw AD and BC. Prove that $\triangle ADP \sim \triangle BPC$.

EXERCISES

1. In Fig. 374, if $BA = 7$, $AP = 21$, and $PC = 28$, find CD.

2. In Fig. 374, if $PB = 32$, $CD = 10$, and $CP = 25$, find BA and AP.

377. Theorem. *If from a point outside a circle a tangent and a secant terminating on the circle are drawn, the product of the secant and its external segment is equal to the square of the tangent.*

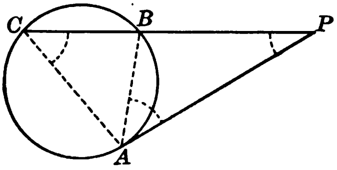


FIG. 375

SUGGESTION. Draw AC and AB. First prove by means of similar triangles that $\frac{CP}{AP} = \frac{AP}{PB}$.

EXERCISES

1. In Fig. 375, if $CB = 5$ and $BP = 8$, find AP.

2. In Fig. 375, if $AP = 10$ and $CP = 15$, find CB and BP.

3. Construct a mean proportional between two given line segments by using the principle of Art. 377.

4. A practical application of the theorem of Art. 377 is made in finding the maximum approximate distance at which an object (for example, a light in a lighthouse of known height) is visible at sea. Let h (Fig. 376) represent the height of the object above the surface of the earth, and let d represent the distance from the object to the horizon. Then, given that the diameter of the earth is 8000 mi., it is clear that $\frac{8000 + h}{d} = \frac{d}{h}$, from which $d^2 = h\,(8000 + h)$. But for all practical purposes h is negligible as compared with the diameter of the earth; accordingly, neglecting h inside the parentheses, we have $d^2 = 8000\,h$, or $h = \frac{d^2}{8000}$, where h and d are given in miles. How far can one see from a ranger's watch tower which is 90 ft. high?

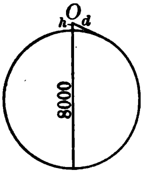


FIG. 376

5. The light in the top of a lighthouse is 80 ft. above the level of the ocean. How far away can the light be seen?

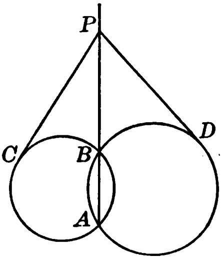


FIG. 377

6. If two tangents are drawn to two intersecting circles from any point outside the circles on the common chord, the tangents are equal (see Fig. 377).

7. In Fig. 378 the perpendicular CD is drawn from the vertex of the rt. $\angle C$ to the hypotenuse AB. Prove that

(a) $\triangle ACD \backsim \triangle CDB$.
(b) $\triangle ACD \backsim \triangle ABC$.
(c) $\triangle CDB \backsim \triangle ABC$.

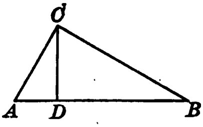


FIG. 378

378. Theorem. *If the ratio of two sides of one triangle equals the ratio of two homologous sides of another triangle, and the angles included between these sides are equal, the triangles are similar.*

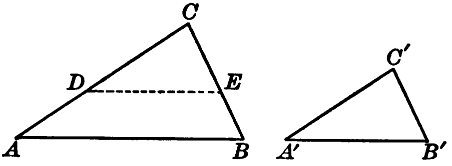


Fig. 379

Given the triangles ABC and $A'B'C'$, with $\frac{C'A'}{CA}$ equal to $\frac{C'B'}{CB}$ and angle C equal to angle C'.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

STATEMENTS	REASONS
1. Lay off $CD = C'A'$ and $CE = C'B'$, and draw DE.	1. Obviously this can be done.
2. Then $\triangle CDE \cong \triangle C'A'B'$.	2. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
3. $\frac{C'A'}{CA} = \frac{C'B'}{CB}$.	3. Given.
4. $\therefore \frac{CD}{CA} = \frac{CE}{CB}$.	4. By substituting CD for $C'A'$ and CE for $C'B'$.
5. $\therefore DE \parallel AB$.	5. If two intersecting lines are cut by two or more lines so that the corresponding segments are in proportion, the lines are parallel.
6. $\therefore \angle CDE = \angle CAB$, and $\angle CED = \angle CBA$.	6. If two parallel lines are cut by two transversals, the corresponding angles are equal.

Proof (*Continued*)

STATEMENTS	REASONS
7. $\therefore \triangle CDE \sim \triangle ABC$.	7. If two angles of one triangle are equal respectively to two angles of another triangle, the triangles are similar.
8. But $\triangle CDE \cong \triangle A'B'C'$.	8. Proved in 2, above.
9. $\therefore \triangle A'B'C' \sim \triangle ABC$.	9. By substitution.

379. Theorem. *Two triangles are similar if the homologous sides are proportional.*

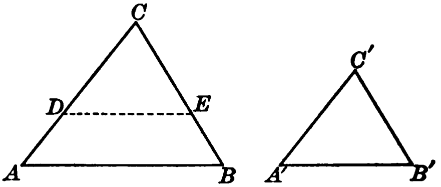


FIG. 380

Given the triangles *ABC* and *A′B′C′*, with

$$\frac{A'C'}{AC} = \frac{C'B'}{CB} = \frac{A'B'}{AB}.$$

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof

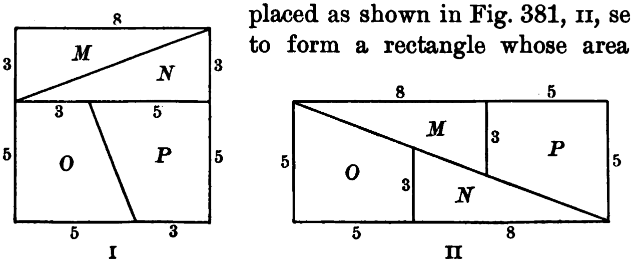
STATEMENTS	REASONS
1. Lay off $CD = C'A'$ and $CE = C'B'$, and draw DE.	1. Obviously this can be done.
2. $\dfrac{A'C'}{AC} = \dfrac{C'B'}{CB}$.	2. Given.
3. Then $\dfrac{CD}{AC} = \dfrac{CE}{CB}$.	3. By substituting CD for $C'A'$ and CE for $C'B'$.
4. Also, $\angle C$ is common to the $\triangle$ CDE and ABC.	4. Obvious.

Proof (*Continued*)

STATEMENTS	REASONS
5. $\therefore \triangle CDE \sim \triangle ABC$.	5. If the ratio of two sides of one triangle equals the ratio of two homologous sides of another triangle and the angles included between these sides are equal, the triangles are similar.
6. $\therefore \frac{DE}{AB} = \frac{CE}{BC}$.	6. Homologous sides of similar triangles are in proportion.
7. But $\frac{A'B'}{AB} = \frac{B'C'}{BC}$.	7. Given.
8. And $B'C' = CE$.	8. By construction.
9. $\therefore \frac{DE}{AB} = \frac{A'B'}{AB}$.	9. Things equal to the same or equal things are equal to each other.
10. $\therefore DE = A'B'$.	10. Obvious.
11. $\therefore \triangle CDE \cong \triangle A'B'C'$.	11. If three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent.
12. $\therefore \triangle ABC \sim \triangle A'B'C'$.	12. By substitution.

EXERCISES

1. A piece of cardboard 8 in. square is cut into four pieces *M*, *N*, *O*, and *P*, as shown in Fig. 381, I. These pieces when placed as shown in Fig. 381, II, seem to form a rectangle whose area is 65 sq. in. Explain the fallacy by means of similar triangles.



I II

FIG. 381

2. Cut out a square like Fig. 381, I, from squared paper. Rearrange the parts as shown in Fig. 381, II, and see if you can see why the extra square unit seems to appear.

380. Practical uses of proportional line segments. We shall now consider a few practical uses of proportional line segments. Fig. 382 shows a pair of *proportional dividers*, an instrument often used by draftsmen when enlarging or reducing drawings. The instrument consists of two graduated bars AD and BC, each carrying a slot in which the clamping screw S slides. When the dividers are adjusted, the lengths CS, DS, BS, and AS are proportional.

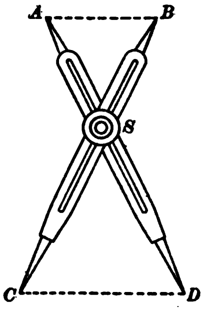


FIG. 382

EXERCISES

1. In Fig. 382 prove that $\frac{CD}{AB} = \frac{CS}{SB}$. If $CS = 8$ in. and $SB = 2$ in., how many times the distance AB is the distance CD?

2. Prove that the common external tangent CB (Fig. 383) to two circles O_1 and O_2, which are tangent externally, is a mean proportional between the diameters of the circles.

SUGGESTION. Prove that $CD = DB = DA$, and that either of these is a mean proportional between O_1A and AO_2.

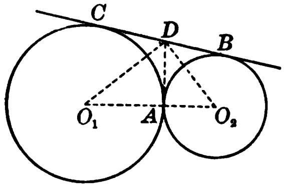


FIG. 383

381. Diagonal scale. Fig. 384 represents a *diagonal* (or *Gunther*) *scale*, which is used by draftsmen for determining the distance between two points accurately to hundredths of an inch. The scale is used only in measuring short lengths, as the distance to be measured is transferred to the scale with a pair of dividers (or compasses if dividers are not available). The distance from 0 to 10 along the bottom of the scale at the left is 1 in.

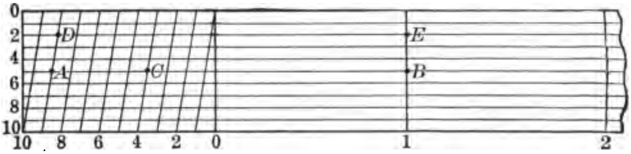


FIG. 384. DIAGONAL SCALE

EXERCISES

1. In Fig. 384 what is the length of a line from line 1 (containing the points *B* and *E*) to the foot of diagonal line 2? from line 1 to the foot of diagonal line 7?

2. In Fig. 384 what is the length of *BA*? of *CA*? of *ED*?

3. Explain the principle which underlies the diagonal scale, and show how to measure lines of the following lengths: 0.35 in., 0.75 in., 1.15 in., 1.85 in.

4. Using the diagonal scale and the compasses, find the hypotenuse of a right triangle whose legs are 1 in. and 2 in. Verify your results by actual computation.

5. Draw a square 1 in. on a side. By means of the compasses and the diagonal scale, find the length of the diagonal of the square to the nearest hundredth of an inch. Check your result by actual computation.

6. By means of the diagonal scale and the compasses, find the altitude of an equilateral triangle 1.32 in. on a side.

382. Pantograph. Another instrument used in enlarging or reducing figures is called a *pantograph.* One form of the pantograph is shown in Fig. 385. By its use a plane figure can be drawn similar to any given figure. The instrument consists of four bars pivoted at Q, P, S, and R to form the parallelogram $QPSR$. The point O is fixed, being provided with a sharp point. The bars are adjustable at

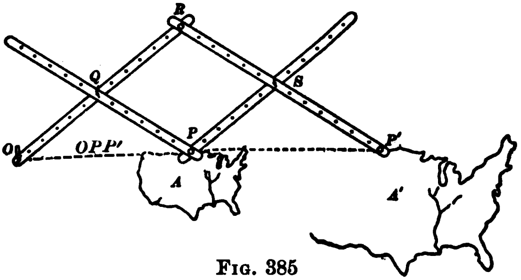


FIG. 385

Q and S so that any desired ratio may be used. Then, if the ratio $\frac{OQ}{OR}$ is made equal to the ratio $\frac{RS}{RP'}$, the points O, P, and P' will lie in the same straight line, and $\frac{OP}{OP'}$ will always equal $\frac{OQ}{OR}$. Thus, if the tracing point P is moved over the outline of a figure to be enlarged, the pencil point P' will trace out a similar figure, as shown. To reduce a figure the tracing point and pencil point may be interchanged.

EXERCISES

1. In Fig. 385 prove that $\triangle OQP \sim \triangle ORP'$.

2. A method used in early times for determining an inaccessible distance is illustrated in Fig. 386. A square like an ordinary carpenter's square was pivoted at the top of a vertical staff. To measure the distance AB the staff was held at A, the leg CF was pointed toward B, and a point D on the ground was determined by the leg CE and marked with a stake. Then AB was found by measuring AC and AD. Show that the method gave the correct result.

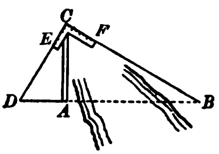


FIG. 386

3. The principle of a photographer's camera depends upon proportional relations. If rays of light from the object AB (Fig. 387) pass through a small opening O and are intercepted by the screen π, an inverted image of the object appears on the screen. If the height of the object AB is 6 ft., and it stands 12 ft. in front of the opening at O, at what distance behind O must the screen be placed so that the image shall be 6 in. high?

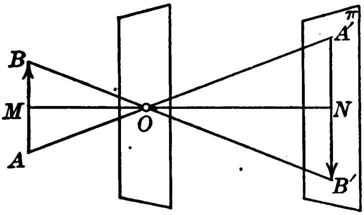


FIG. 387

4. In Fig. 387, if $MO = 20$ ft., $ON = 6$ ft., and $AB = 30$ ft., find the length of $A'B'$.

5. In Fig. 387, if $MO = 30$ ft., $ON = 40$ ft., and $A'B' = 32$ ft., what is the length of AB?

6. In Fig. 387 state the conditions under which the image will be of the same length as the object. Show that your answer is correct.

7. In Fig. 388 the height of the right circular cone is 12 cm., and the diameter of its base is 8 cm. What is the diameter of the circular section made by a plane 5 cm. above the base?

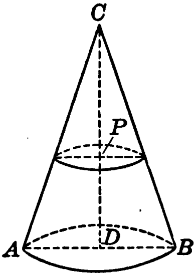


FIG. 388

8. The strongest rectangular beam that can be cut from a cylindric log has a cross section as shown in Fig. 389. This cross section is determined as follows: The circle is allowed to represent the log, and the diameter AC is divided into three equal parts, AE, EF, and FC. The perpendiculars FD and EB are drawn to AC, and the points A, B, C, D are connected. Prove that $ABCD$ is a rectangle.

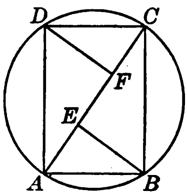


FIG. 389

383. Theorem. *In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, then*

1. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

2. *Either side about the right angle is a mean proportional between the hypotenuse and the segment of the hypotenuse adjacent to that side.*

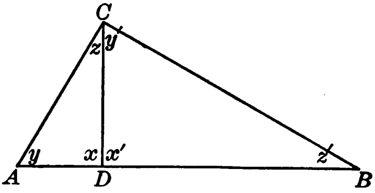


FIG. 390

Proof. The proof is left to the student.

384. Corollary. *If a perpendicular is drawn from any point on a circle to a diameter, then*

1. *The perpendicular is the mean proportional between the segments of the diameter.*

2. *The chord joining the point to either end of the diameter is the mean proportional between the whole diameter and the segment of it adjacent to the chord.*

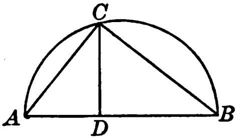


FIG. 391

EXERCISES

1. Find the altitude drawn to the hypotenuse of a right triangle if the altitude divides the hypotenuse into segments whose lengths are 5 and 15 respectively. Find also each leg of the right triangle.

2. In Fig. 391, if $CD = 10$ and $AB = 14$, find AD and DB.

3. A chord of a circle is 18 in. in length, and the midpoint of the chord is 4 in. from the midpoint of its arc. What is the diameter of the circle?

385. ***Construction problem.*** *To construct a mean proportional between two given line segments.*

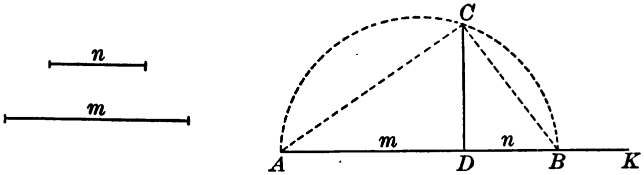


FIG. 392

Given the line segments *m* and *n*.

Required *to construct a mean proportional between m and n.*

Construction. On a working line AK (Fig. 392) lay off m from A to D and n from D to B. Bisect AB, and with AB as a diameter construct a semicircle as shown. At D erect a perpendicular intersecting the circle at C.

Then CD is the required mean proportional.

Proof. The proof is left to the student.

EXERCISES

1. Construct a mean proportional between 9 and 16; between 25 and 36.

2. Construct a line segment equal to $a\sqrt{2}$, where a is any line segment.

Analysis. Let $x = a\sqrt{2}.$

Then $x^2 = 2a^2,$

by squaring both members of the equation,

and $x \cdot x = 2a \cdot a.$

Therefore $\frac{2a}{x} = \frac{x}{a}.$

Thus it is seen that x is simply a mean proportional between $2a$ and a.

Construction. The construction is left to the student.

3. Construct a line segment equal to $\sqrt{5mn}$, where m and n are any two line segments.

4. Show that when a mean proportional between two line segments is found, the geometrical square root of their product is found.

5. Find geometrically $\sqrt{27}$; $\sqrt{31}$.

6. Show by the theorem of Art. 377 that in Fig. 375 the tangent AP is a mean proportional between the secant CP and its external segment PB.

7. Construct a $\triangle A'B'C'$ similar to a given $\triangle ABC$.

8. Construct a circle that shall be tangent to a given line and shall pass through two given points. Discuss all the possibilities.

SUGGESTION. Use the analytic method. Imagine the construction done and thus try to discuss how the circle may be drawn.

386. *Theorem.* *Two polygons are similar if they are composed of the same number of triangles similar each to each and similarly placed.*

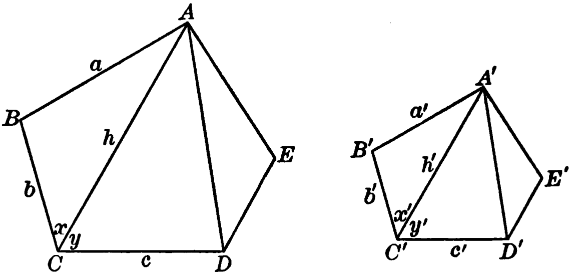


FIG. 393

Given the polygons *ABCD* ⋯ and *A'B'C'D'* ⋯, composed of the triangles *ABC*, *ACD*, ⋯, similar to the triangles *A'B'C'*, *A'C'D'*, ⋯, and similarly placed.

To prove that $ABCD \cdots \sim A'B'C'D' \cdots$.

Proof

STATEMENTS	REASONS
1. $\angle B = \angle B'$, and $\angle x = \angle x'$.	1. Given.
2. Also, $\angle y = \angle y'$.	2. Given.
3. $\therefore \angle C = \angle C'$.	3. Equals added to equals give equals.
4. In like manner, $\angle D = \angle D'$, and so on.	4. For similar reasons.
5. $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.	5. By definition of similar triangles.
6. But $\dfrac{AC}{A'C'} = \dfrac{CD}{C'D'} = \dfrac{AD}{A'D'} = \cdots$.	6. For the same reason.
7. $\therefore \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \cdots$.	7. Things equal to the same thing are equal to each other.
8. $\therefore ABCD \cdots \sim A'B'C'D' \cdots$.	8. Because they are mutually equiangular and their homologous sides are in proportion.

387. Theorem. *Homologous diagonals divide similar polygons into the same number of triangles, similar each to each and similarly placed.*

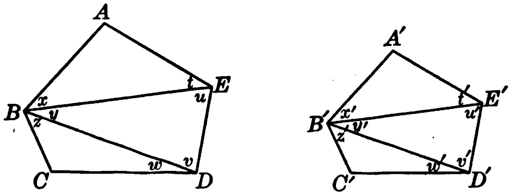


FIG. 394

Given the similar polygons $ABCD \cdots$ **and** $A'B'C'D' \cdots$.

To prove that $\triangle BCD \sim \triangle B'C'D'$, $\triangle BDE \sim \triangle B'D'E'$, $\cdots$.

Proof. The proof is left to the student.

388. Construction problem. *Upon a given line segment homologous to a given side of a given polygon, construct a polygon similar to the given polygon.*

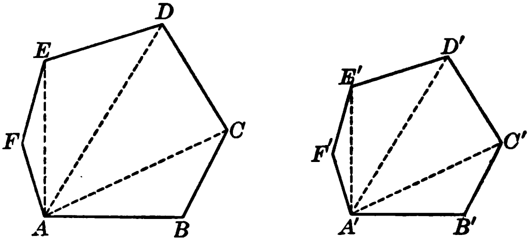


FIG. 395

Given the line *A'B'* homologous to the line *AB* of the polygon *ABCDEF*.

Required *to construct on $A'B'$ a polygon similar to the polygon $ABCDEF$.*

SUGGESTION. Draw the diagonals AC, AD, and AE. Then on $A'B'$ construct $\triangle A'B'C'$ similar to $\triangle ABC$. On $A'C'$, homologous to AC, construct $\triangle A'C'D'$ similar to $\triangle ACD$, and so on.

389. Theorem. *The perimeters of similar triangles have the same ratio as any two homologous sides.*

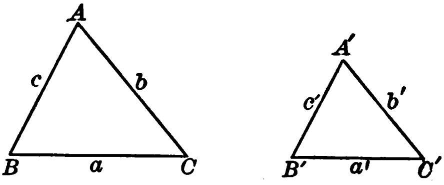


FIG. 396

Given the similar triangles *ABC* and *A'B'C'*.

To prove that $\frac{a+b+c}{a'+b'+c'}=\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$.

Proof. The proof is left to the student.

EXERCISES

1. The perimeter of a certain triangle is 22 cm., and the sides of a similar triangle are 5.5 cm., 3.8 cm., and 8.2 cm. respectively. Find the lengths of each side of the first triangle.

2. The perimeters of two similar triangles are $x^2 + 1$ and $2x$, and a pair of homologous sides are 5 and 4 respectively. Find the value of x.

390. *Theorem.* *The perimeters of two similar polygons have the same ratio as any two homologous sides.*

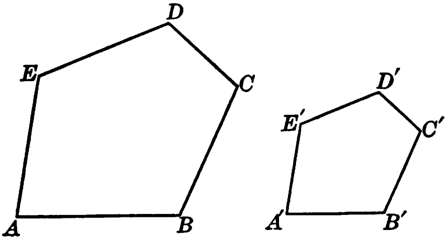


FIG. 397

Given the similar polygons *ABCDE* and *A′B′C′D′E′*.

To prove that

$$\frac{AB+BC+CD+DE+EA}{A'B'+B'C'+C'D'+D'E'+E'A'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\cdots.$$

Proof. The proof is left to the student.

EXERCISES

1. The base of a triangle is 64 in., and the altitude is 14 in. Find the length of the upper base of a trapezoid formed by a line parallel to the base of the triangle and 7 in. above it.

2. The perimeters of two triangles are $2x^2 + 4x + 6$ and $x^2 + 2x + 3$, and a pair of homologous sides are 8 and 4 respectively. Are the triangles similar?

3. The perimeters of two similar polygons are 120 and 72. If a side of the first is 25, find a homologous side of the second.

CHAPTER VIII

TRIGONOMETRY

391. Scale drawings; indirect measurements. From the preceding work every student is familiar with the principle of scale drawings, such as are represented by blueprints, shop drawings, architect's plans, and geography and railroad maps, and knows that in such drawings the actual dimensions of the object drawn are represented by proportional line segments which show the relative sizes of the actual dimensions. Hence from a scale drawing of an object we are able to determine dimensions of the object which are not given in the drawing, or we can obtain measurements of distances which in the real object it is difficult or impossible to measure directly.

This last procedure illustrates a practical use which is often made of scale drawings in such cases as measuring (1) the heights of towers, buildings, or trees; (2) the width of ponds, lakes, or rivers; (3) the length of boundary lines which pass through houses, barns, or other obstructions. Obviously, in each of these cases the use of the steel tape to measure the distance directly is either difficult or impossible.

We can usually determine such distances by the following method:

1. Measure enough actual lines and angles in the figure so that a scale drawing of the figure can be made.

2. Draw the figure to scale, preferably on squared paper. This is always a good check on your computation.

3. With the compasses and squared paper, measure the lines in the drawing which represent the inaccessible lines of the actual figure.

4. Compute from the measurements obtained in 3 the actual dimension, expressing it in the units used in measuring the lines of the actual figure.

392. Limitations of scale drawings. Although scale drawings are useful, the student will readily appreciate the fact that they have their limitations. Consider a map in which 1 mm. represents 0.1 mi., and you will see how a slight inaccuracy in measuring the map would mean a large error when translated into the distance in feet on the ground. Thus, you would hardly agree to buy a building lot in the heart of a city if the area of the lot had been determined only by a scale drawing. Scale drawings, however, are used extensively by surveyors and engineers in the following ways: (1) as a method of estimating probable results; (2) as a help to clear thinking about the relations of lines and angles involved in a geometric drawing; and (3) as a valuable check on results obtained by more accurate methods. It is such use as this that the student is expected to make of scale drawings in the work of this chapter.

The student will see that in determining the lengths of lines and the sizes of angles we need a more refined method than is afforded by scale drawings, especially where a high degree of accuracy is required. We shall proceed to consider a far more powerful and efficient method of determining such lines and angles, known as *trigonometry*, which means "triangle-measure." Most students will find this subject fascinating, because the method of solution is simple and the results obtained are as accurate as the measurements of the lines and angles which are directly measured.

393. Similar right triangles. A few exercises on similar right triangles will help the student to understand the new method of determining lines and angles. This method may be used independent of scale drawings, is generally shorter, and is the foundation of much future mathematical work.

EXERCISES

1. With the protractor construct a right triangle having an angle of 37°, and letter the figure as shown in Fig. 398. Measure the lines a, b, and c. Find the value of the ratios $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a}{b}$ to two decimal places.

2. Compare your result with those of other members of the class. Did all members of the class use the same length for the bases? Are any two of the triangles necessarily of the same size? Show why your result for $\frac{a}{c}$ should be the same as the results of your classmates.

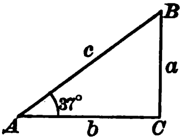


FIG. 398

3. *If two right triangles have an acute angle of one equal to an acute angle of the other, the ratios of their corresponding sides are equal.*

SUGGESTION. Use Fig. 399 and write two proportions.

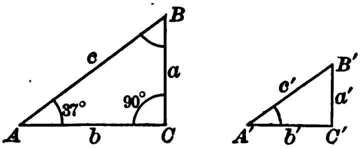


FIG. 399

4. Could you draw a right triangle with $\angle A = 37°$ in which $\frac{a}{c}$ does not equal approximately 0.60, or $\frac{3}{5}$? Prove your answer.

HINT. The fact that $\frac{a}{c} = \frac{3}{5}$ means that in every right triangle the side opposite an angle of 37° is approximately $\frac{3}{5}$ as long as the hypotenuse.

5. A balloon B (Fig. 400) is fastened by a cable 200 ft. long. Owing to the wind the cable is held practically straight and makes an angle of 37° with the horizontal. How high is the balloon?

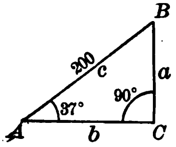


FIG. 400

Solution. The $\triangle ABC$ (Fig. 400) is similar to every triangle drawn by the class in Ex. 1. Prove that this is true.

Therefore $\frac{a}{200} = 0.60.$ Why?

Solving the above equation,

$$a = 120 \text{ ft.}$$

Notice that the solution is exceedingly simple and that the accuracy of the result does not depend upon the accuracy of the figure.

394. Sine of an angle. The ratio $\frac{a}{c}$ (Fig. 401) is called the *sine* of the $\angle A$. The abbreviation for "sine" is "sin." This definition may be written $\sin A = \frac{a}{c}$. Thus, from Ex. 1, Art. 393, we see that $\sin 37° = \frac{3}{5} = 0.60$ (approximately). Do you think we would have obtained the same value for $\frac{a}{c}$ in Ex. 1, Art. 393, if we had made the angle one of 47° instead of one of 37°?

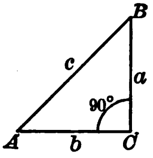


FIG. 401

EXERCISES

1. Using the definition given in Art. 394, find the sine of 20°.

HINT. As in Ex. 1, Art. 393, construct the triangle, measure a and c, and find the value of $\frac{a}{c}$ to two decimal places.

2. Find the sine of each of the following angles: 10°; 15°; 25°; 32°; 47°; 68°; 87°. Compare each result with the results obtained by your classmates.

395. Table of sines. The preceding exercises show that the sine of the angle changes with the angle; that is, sin 68° is not equal to sin 37°. By taking a large sheet of graph paper and a large unit of length we could make a fairly good table of sines of different angles, but it would be too much trouble to do this for every problem. Such a table has been very carefully calculated for you in the first column of the table on page 315.

EXERCISE

Turn to the table on page 315 and see how efficient you have been by comparing your results for Ex. 2, Art. 394, with the table.

396. Cosine of an angle. The exercises given below introduce another trigonometric ratio.

EXERCISES

1. Construct a right triangle with $\angle A = 43°$ (see Fig. 401). Measure b and c, and find the quotient $\frac{b}{c}$ to two decimal places. Compare your result with those of the other members of the class.

2. Show that all results in Ex. 1 should agree. The ratio $\frac{b}{c}$ (Fig. 401) is called the *cosine* of the $\angle A$. In the case of an angle of 43° in a right triangle it means that the *side adjacent to the angle* is about $\frac{73}{100}$ of the hypotenuse. The abbreviation for "cosine" is "cos."

3. Find the cosine of 5°; 18°; 25°; 35°; 47°; 65°; 87°.

4. Compare the results in Ex. 3 with the table on page 315.

397. Tangent of an angle. We shall now introduce a third important trigonometric ratio. In many ways this ratio is the most interesting and valuable of the three ratios which we discuss in this chapter.

Historically this third ratio was discovered first.

EXERCISES

1. In Fig. 401 what is the value of $\frac{a}{b}$ to two decimal places? Compare your result with those obtained by other members of the class.

2. Show that all the results obtained for $\frac{a}{b}$ in Ex. 1 should agree. The ratio $\frac{a}{b}$ is called the *tangent* of the $\angle A$. In speaking of the tangent of 43° we mean that the side opposite $\angle A$ is $\frac{93}{100}$ (approximately) of the side adjacent to the angle. The abbreviation for "tangent" is "tan."

3. Find the tangent of 11°; 36°; 45°; 57°; 82°.

4. Compare the results in Ex. 3 with the table of tangents on page 315.

398. Trigonometric ratios; solving the triangle. The ratios $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a}{b}$ are called *trigonometric ratios.* We shall now proceed to show that the use of these ratios greatly simplifies the solution of many problems which involve indirect measurement. By their use any part of a right triangle can be found if two parts (not both angles) besides the right angle are given. This process is called *solving the triangle.*

399. Summary. The following outline will be found convenient to the student in helping him to remember the definitions (see Fig. 402) of the trigonometric ratios:

1. $\sin A = \frac{a}{c} = \frac{\textit{side opposite}}{\textit{hypotenuse}}$.
2. $\cos A = \frac{b}{c} = \frac{\textit{side adjacent}}{\textit{hypotenuse}}$.
3. $\tan A = \frac{a}{b} = \frac{\textit{side opposite}}{\textit{side adjacent}}$.

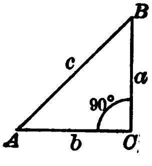


FIG. 402

400. Trigonometric ratios as functions. Either by your own efforts to make a table of ratios, or by a study of the table given on page 315, it is easy to see that the value of the ratio changes as the angle changes; that is, a trigonometric ratio depends for its value upon the size of the angle. Hence the ratios furnish us with yet another clear example of the function idea. We therefore refer to them as *trigonometric functions.*

401. Table of trigonometric ratios. The student should now become thoroughly familiar with the table of trigonometric ratio which is given on the following page. The table gives to three decimal places the value of the sine, cosine, and tangent for angles from 1° to 89°. The values given are in most cases only approximate, but are accurate enough for all ordinary work.

The problems, like those given on page 323, are intended to furnish the student practice in the use of the table.

402. Construction of an acute angle. If we know the value of a trigonometric function, it is easy to construct the angle and to find the values of its other trigonometric functions. The method will now be illustrated by the two following examples:

1. If A is an acute angle and $\sin A = \frac{2}{3}$, construct $\angle A$ and find the values of its cosine and tangent.

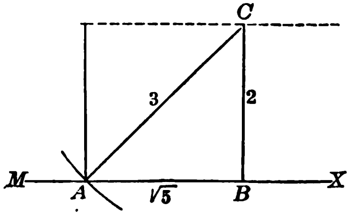


FIG. 403

Solution. If $\sin A = \frac{2}{3}$, the side opposite $\angle A$ must be 2 units in length and the hypotenuse must be 3 units in length. Hence on a working line MX draw $CB \perp MX$ and make CB 2 units in length. With C as a center and a radius of 3 units draw an arc cutting MX at some point A. Draw AC. Then $\angle BAC$ is the required angle,

Angle	**Sine** $\left(\frac{\text{opp.}}{\text{hyp.}}\right)$	**Cosine** $\left(\frac{\text{adj.}}{\text{hyp.}}\right)$	**Tangent** $\left(\frac{\text{opp.}}{\text{adj.}}\right)$	**Angle**	**Sine** $\left(\frac{\text{opp.}}{\text{hyp.}}\right)$	**Cosine** $\left(\frac{\text{adj.}}{\text{hyp.}}\right)$	**Tangent** $\left(\frac{\text{opp.}}{\text{adj.}}\right)$
0°	.000	1.000	.000	**45°**	.707	.707	1.000
1°	.017	1.000	.017	**46°**	.719	.695	1.036
2°	.035	.999	.035	**47°**	.731	.682	1.072
3°	.052	.999	.052	**48°**	.743	.669	1.111
4°	.070	.998	.070	**49°**	.755	.656	1.150
5°	.087	.996	.087	**50°**	.766	.643	1.192
6°	.105	.995	.105	**51°**	.777	.629	1.235
7°	.122	.993	.123	**52°**	.788	.616	1.280
8°	.139	.990	.141	**53°**	.799	.602	1.327
9°	.156	.988	.158	**54°**	.809	.588	1.376
10°	.174	.985	.176	**55°**	.819	.574	1.428
11°	.191	.982	.194	**56°**	.829	.559	1.483
12°	.208	.978	.213	**57°**	.839	.545	1.540
13°	.225	.974	.231	**58°**	.848	.530	1.600
14°	.242	.970	.249	**59°**	.857	.515	1.664
15°	.259	.966	.268	**60°**	.866	.500	1.732
16°	.276	.961	.287	**61°**	.875	.485	1.804
17°	.292	.956	.306	**62°**	.883	.469	1.881
18°	.309	.951	.325	**63°**	.891	.454	1.963
19°	.326	.946	.344	**64°**	.899	.438	2.050
20°	.342	.940	.364	**65°**	.906	.423	2.145
21°	.358	.934	.384	**66°**	.914	.407	2.246
22°	.375	.927	.404	**67°**	.921	.391	2.356
23°	.391	.921	.424	**68°**	.927	.375	2.475
24°	.407	.914	.445	**69°**	.934	.358	2.605
25°	.423	.906	.466	**70°**	.940	.342	2.747
26°	.438	.899	.488	**71°**	.946	.326	2.904
27°	.454	.891	.510	**72°**	.951	.309	3.078
28°	.469	.883	.532	**73°**	.956	.292	3.271
29°	.485	.875	.554	**74°**	.961	.276	3.487
30°	.500	.866	.577	**75°**	.966	.259	3.732
31°	.515	.857	.601	**76°**	.970	.242	4.011
32°	.530	.848	.625	**77°**	.974	.225	4.331
33°	.545	.839	.649	**78°**	.978	.208	4.705
34°	.559	.829	.675	**79°**	.982	.191	5.145
35°	.574	.819	.700	**80°**	.985	.174	5.671
36°	.588	.809	.727	**81°**	.988	.156	6.314
37°	.602	.799	.754	**82°**	.990	.139	7.115
38°	.616	.788	.781	**83°**	.993	.122	8.144
39°	.629	.777	.810	**84°**	.995	.105	9.514
40°	.643	.766	.839	**85°**	.996	.087	11.430
41°	.656	.755	.869	**86°**	.998	.070	14.301
42°	.669	.743	.900	**87°**	.999	.052	19.081
43°	.682	.731	.933	**88°**	.999	.035	28.636
44°	.695	.719	.966	**89°**	1.000	.017	57.290
45°	.707	.707	1.000	**90°**	1.000	.000	

Note. The abbreviation *hyp.* means "hypotenuse"; *adj.* means "the side adjacent to the angle"; *opp.* means "the side opposite the angle."

because its sine is equal to $\frac{2}{3}$ by construction. By measurement $\angle A$ is seen to be approximately 42°.

Using the Pythagorean theorem, we have

$$\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2,$$

or

$$\overline{AC}^2 = 9 - 4 = 5.$$

$$\therefore AC = \sqrt{5}.$$

$$\therefore \cos A = \frac{\sqrt{5}}{3} \text{ (by definition of "cosine")}$$

and

$$\tan A = \frac{2}{\sqrt{5}} = \frac{\sqrt{10}}{5} \text{ (by definition of "tangent").}$$

2. Given $\cos A = \frac{3}{5}$, construct $\angle A$ and find the values of its sine and tangent.

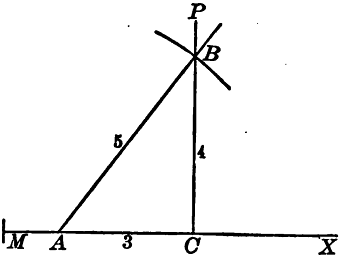


FIG. 404

Solution. If $\cos A = \frac{3}{5}$, the side adjacent to $\angle A$ must be 3 units in length and the hypotenuse must be 5 units in length. Hence we take a working line MX and on it lay off $AC = 3$ units. At C erect the perpendicular CP, and with A as a center and a radius of 5 units draw an arc cutting CP at B. Draw AB. Then $\angle BAC$ is the required angle, because $\cos A = \frac{3}{5}$ by construction. By measurement $\angle A$ is found to be approximately 53°.

Then

$$\overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2,$$

or

$$\overline{BC}^2 = 25 - 9 = 16.$$

$$\therefore BC = 4.$$

$$\therefore \sin A = \frac{4}{5} \text{ (by definition of "sine")}$$

and

$$\tan A = \frac{4}{3} \text{ (by definition of "tangent").}$$

NOTE. The use of squared paper in cases similar to the problems above will save time and insure accuracy.

EXERCISES

From the following data construct the acute $\angle A$, then find the size of the angle (as accurately as possible) and the values of the functions which are not given (sine, cosine, or tangent):

1. $\sin A = \frac{1}{2}$.
2. $\cos A = \frac{4}{5}$.
3. $\tan A = \frac{3}{4}$.
4. $\sin A = \frac{4}{5}$.
5. $\cos A = \frac{2}{3}$.
6. $\tan A = \frac{3}{1}$.
7. $\tan A = 1$.
8. $\sin A = \frac{3}{4}$.
9. $\cos A = \frac{3}{5}$.

403. Functions of 30° and 60°. If the student solved Ex. 1, Art. 402, carefully, he found that

$$\sin 30° = \tfrac{1}{2},$$

$$\cos 30° = \frac{\sqrt{3}}{2},$$

and

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

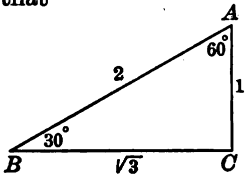


FIG. 405

These values can be verified by a study of Fig. 405, and the same figure shows that

$$\sin 60° = \frac{\sqrt{3}}{2},$$

$$\cos 60° = \tfrac{1}{2},$$

and

$$\tan 60° = \sqrt{3}.$$

404. Functions of 45°. If the student solved Ex. 7, Art. 402, carefully, he found that

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

and

$$\tan 45° = 1.$$

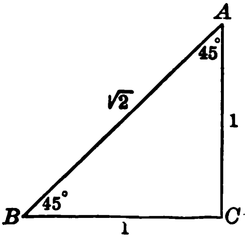


FIG. 406

These values can be verified by studying Fig. 406.

405. Rationalizing the denominator of a fraction. In Arts. 403 and 404, $\frac{1}{\sqrt{2}}$ was written as $\frac{\sqrt{2}}{2}$ because we wished to make the number easier to use. That it is easier to divide $\sqrt{2}$ by 2 than to divide 1 by $\sqrt{2}$ the student will readily see. If we multiply $\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{2}}{\sqrt{2}}$ we have $\frac{\sqrt{2}}{2}$: the value of our original fraction $\frac{1}{\sqrt{2}}$ is not changed and we have a rational number in the denominator. This process of changing $\frac{1}{\sqrt{2}}$ to $\frac{\sqrt{2}}{2}$ is called *rationalizing the denominator.*

EXERCISES

Rationalize the denominators of the following fractions:

1. $\frac{1}{\sqrt{3}}$.
2. $\frac{\sqrt{3}}{\sqrt{2}}$.
3. $\frac{5}{\sqrt{a}}$.
4. $\frac{8}{3\sqrt{2}}$.
5. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{5}}$.
6. $\frac{\sqrt{5}-\sqrt{3}}{2\sqrt{2}}$.
7. $\frac{\sqrt{10}-\sqrt{5}}{3\sqrt{6}}$.
8. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}$.
9. $\frac{\sqrt{c}-\sqrt{d}}{\sqrt{a}}$.
10. $\frac{2}{3-\sqrt{2}}$.

Solution. In Ex. 10 multiply $\frac{2}{3-\sqrt{2}}$ by $\frac{3+\sqrt{2}}{3+\sqrt{2}}$; that is, multiply both numerator and denominator by $3+\sqrt{2}$. Then we have

$$\frac{2}{3-\sqrt{2}}=\frac{2(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}=\frac{6+2\sqrt{2}}{9-2}=\frac{6+2\sqrt{2}}{7}.$$

11. $\frac{4}{2+\sqrt{3}}$.
12. $\frac{5}{3-\sqrt{6}}$.
13. $\frac{4}{\sqrt{2}+3}$.
14. $\frac{2-\sqrt{3}}{5+\sqrt{2}}$.
15. $\frac{1+\sqrt{2}}{2-\sqrt{6}}$.
16. $\frac{5+\sqrt{3}}{5-\sqrt{3}}$.

17. Solve for x the equation $3x = \sqrt{2}(1 + 4x)$ and express the result as a fraction with a rational denominator.

Solution.

$$3x = \sqrt{2} + 4x\sqrt{2}.$$

$$3x - 4x\sqrt{2} = \sqrt{2}.$$

$$x(3 - 4\sqrt{2}) = \sqrt{2}.$$

$$x = \frac{\sqrt{2}}{3 - 4\sqrt{2}} = \frac{\sqrt{2}(3 + 4\sqrt{2})}{(3 - 4\sqrt{2})(3 + 4\sqrt{2})}$$

$$= \frac{3\sqrt{2} + 8}{9 - 32} = \frac{3\sqrt{2} + 8}{-23}.$$

18. Solve the following for x and express the result as a fraction with a rational denominator:

(a) $2x = \sqrt{3}(1 + 3x)$.

(b) $5x = \sqrt{2}(2 + 4x)$.

406. Approximate measurements and computations. The student has already learned much about the practical difficulty of precise measurements and knows that in all actual measurements we obtain only approximate results. The degree of accuracy of our work should first of all be determined by the degree of accuracy required in the result. Thus, it would be foolish, indeed, to carry computations to four or five decimal places when the result we seek need be accurate only to tenths of the unit employed.

The accuracy of our results will depend also, of course, upon the method which we use, the accuracy of the instruments with which we measure, and that of the tables with which we compute. Our results can never have a greater degree of accuracy than that of the least accurate of the measurements or numbers used in the computation. For example, if one number is accurate to three figures and another to four figures, their sum, difference, product, or quotient will not as a general rule be accurate to more than three figures. That is why it is unnecessary and sometimes misleading to use a four-place table when only numbers with three places are found in the data of the problem. But the student should bear in mind that computations must always be mathematically accurate up to the point required, and the fact that we require a result accurate to tenths only is no excuse for careless errors in arithmetical operations.

407. Solving right triangles. We know from our preceding work in geometry that in a right triangle if two other parts besides the right angle are given (one of these at least being a side), the triangle is determined. We have also seen how these triangles may be constructed. Hence in solving right triangles we must know how to handle the two following cases:

1. Given one side and one acute angle.
2. Given any two sides.

The student should convince himself that these two cases cover all the possibilities.

It is desirable in solving a right triangle to have some well-ordered plan of procedure. The following scheme is helpful:

1. Read the statement of the problem carefully.
2. Make an accurate scale drawing which shows the given angles and lines in their correct relation to those that are required (if it is an applied problem it may be necessary to draw helping lines).
3. Examine your drawing carefully and decide upon the simplest way to attack the problem.
4. Write the formulas which you think it is necessary to use.
5. Make your computations accurately, carrying them to the degree of precision demanded by the conditions.

6. Apply some kind of check to your result. Either use the scale drawing as a rough check or check by algebraic methods.

Two simple examples, which illustrate the two cases given above, will now be solved in order that the student may see how to use the preceding plan.

CASE I. *Given one side and an acute angle.*

1. Given $b = 30$ ft., $A = 10°$, find the remaining parts of the rt. $\triangle ABC$.

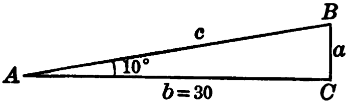


FIG. 407

Construction. Draw the rt. $\triangle ABC$ to scale as shown in Fig. 407.

Solution. $\angle B = 90° - \angle A = 90° - 10° = 80°$.

$\frac{a}{b} = \tan A$ is the formula to use in solving for a,

and $\frac{b}{c} = \cos A$ is the formula to use in solving for c.

Then $\frac{a}{30} = \tan 10°$. But by the table (page 315), $\tan 10° = 0.176$.

$$\therefore \frac{a}{30} = 0.176, \text{ and } a = 30 \times 0.176 = 5.28.$$

Also $\frac{30}{c} = \cos 10° = 0.985$ (by the table, Art. 401).

$$\therefore 30 = 0.985\,c, \text{ and } c = \frac{30}{0.985} = 30.46.$$

Check. We can use as a check the Pythagorean relation $c^2 = a^2 + b^2$. It makes the work easier if we write this relation as

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= (c + b)(c - b). \end{aligned}$$

In ordinary cases, however, it is generally sufficient to use the scale drawing as a check. Measure each of the lines computed and multiply the number of units in each by the number of feet which a unit represents.

NOTE. The student should observe that it is always better to use a function which contains the given rather than the computed values, as the computed values might not be correct.

CASE II. *Given any two sides.*

2. Given $a = 15$ ft., $c = 50$ ft., find the remaining parts of the rt. $\triangle ABC$.

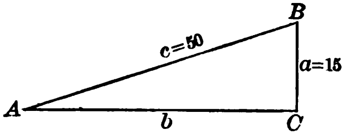


FIG. 408

Construction. Draw the rt. $\triangle ABC$ to scale as shown in Fig. 408.

Solution. Use the following formulas:

$$\sin A = \frac{a}{c}. \tag{1}$$

$$B = 90° - A. \tag{2}$$

$$\frac{b}{c} = \cos A. \tag{3}$$

From formula (1), $\sin A = \frac{15}{50} = 0.3.$

By the table (page 315), $0.3 = \sin 18°$ (approximately).

$$\therefore \angle A = 18°.$$

From formula (2), $\angle B = 90° - 18° = 72°.$

From formula (3), $b = 50 \times 0.951 = 47.55.$

Check. The check by measurement verifies the preceding results approximately.

EXERCISES

From the following data solve the rt. $\triangle ABC$ if

1. $A = 30°$, $a = 30$ ft.
2. $B = 42°$, $b = 60$ ft.
3. $A = 64°$, $b = 22$ ft.
4. $A = 35°$, $b = 85$ ft.
5. $b = 92.5$ ft., $c = 100$ ft.
6. $a = 20$ ft., $c = 35$ ft.
7. $A = 40°$, $c = 80$ ft.
8. $B = 82°$, $c = 100$ ft.

408. Transit. When it is important to secure a higher degree of accuracy than is possible with the instrument described in Art. 38, a surveyor's instrument called a *transit* (Fig. 409) may be used. The essential parts of the transit are (1) a horizontal graduated circle D for measuring angles in the horizontal plane; (2) a graduated circle C for measuring angles in the vertical plane; and (3) a telescope AB for sighting along the sides of the angle. A fuller description of the transit may be found in any textbook on trigonometry or surveying.

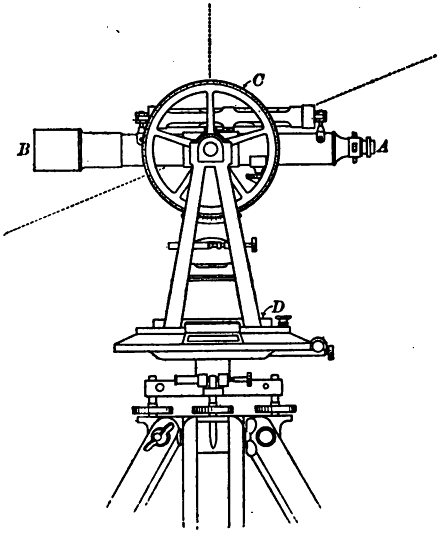


FIG. 409. SURVEYOR'S TRANSIT

409. Angle of elevation. The $\angle HAK$ in Fig. 410 shows what is meant by an *angle of elevation.* To find the angle of elevation the transit is placed at A and the telescope is first pointed horizontally toward the height *KH*, which is being measured. The farther end of the telescope is then raised until *K* is on the line of sight. The $\angle HAK$ through which the telescope turns is the angle of elevation of *K* from *A*, the point of observation.

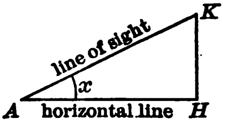


FIG. 410

If the transit is not available for practical work in the field, a simple device like that shown in Fig. 411 may be made for measuring the $\angle KAH$. This device consists of two 1-foot rules or yardsticks riveted together so that they are free to turn on the rivet as a pivot. A plumb line is hung at the pivot as shown, and a protractor is fastened to one of the sticks so that its center is at the rivet. The instrument is used in much the same way as the transit.

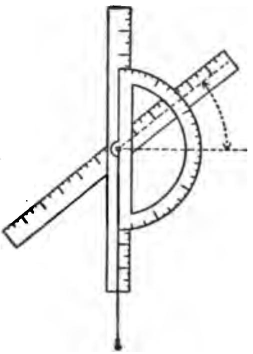


FIG. 411

410. Angle of depression. If a transit at *M* in the top of a lighthouse (Fig. 412) is first pointed horizontally (that is, the reading is 0) and then the farther end is lowered (depressed) until the telescope points to a boat at *B*, the $\angle HMB$ through which the telescope turns is the *angle of depression* of the boat from the point *M*.

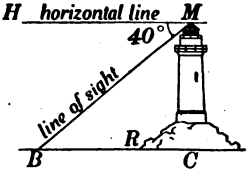


FIG. 412

411. Applied problems. The following set of applied problems gives further practice in solving right triangles. The student should notice that he is not required to solve the triangle completely, as in Art. 407.

EXERCISES

1. A balloon B (Fig. 413) is anchored to the ground at a point A by a rope which makes an angle of 57° with the ground. The point C on the ground directly beneath the balloon is 146 ft. from A. Assuming the rope to be straight, find the height of the balloon.

Solution. Let a = height of balloon.

Then $\frac{a}{146}$ = tangent of 57°.

But by the table,

$$\tan 57° = 1.54.$$

Hence $$\frac{a}{146} = 1.54,$$

or $$a = 224.84 \text{ ft.}$$

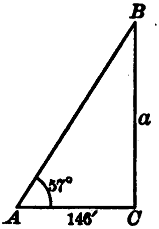


FIG. 413

NOTE. The figure does not need to be drawn accurately, since our results are obtained independently. The solution is brief and depends for its accuracy upon the accuracy of measurement of the $\angle A$ and the line AC and the accuracy of the tangent table.

2. The angle of elevation of an airplane from a point A on level ground is 63°. The point C on the ground directly beneath the airplane is 415 yd. from A. Find the height of the airplane.

3. The length of a kite string is 310 yd., and the angle of elevation of the kite is 38°. Find the height of the kite, supposing the kite string to be stretched straight.

4. A pole 30 ft. in length stands vertically in a level field, and the length of its shadow is 16.78 ft. Find the angle of elevation of the sun.

HINT. Find the value of the tangent of the angle of elevation and then look in the table to see what angle has a tangent corresponding to this value. It may be necessary for you to take an approximate angle, since the table is calculated only for whole degrees. Ask your instructor to show you a more complete table of trigonometric functions.

5. A tree is broken and blown over by the wind so that its two parts form with the ground a right triangle. The upper part makes an angle of 65° with the ground, and the distance on the ground from the trunk to the top of the tree is 47 ft. Find the original height of the tree.

6. A circular pool has a pole standing vertically at its center, and the top of the pole is 55 ft. above the surface of the water. At a point on the edge of the pool the angle of elevation of the top of the pole is 35°. Find the radius and the area of the pool.

7. A ladder 45 ft. long leans against a house and reaches to a point 29.6 ft. from the ground. Find the angle between the ladder and the house and the distance from the foot of the ladder to the house.

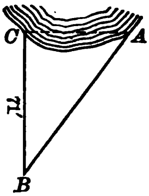


FIG. 414

8. To find the distance across a lake between two points A and C (Fig. 414), a surveyor measured off 71 ft. on a line BC which was perpendicular to AC. He then found that $\angle CAB$ measured 52°. Find AC. Could you also find AB by trigonometry?

9. The Washington Monument is 555 ft. high. How far apart are two observers who from points due west of the monument observe its angles of elevation to be 20° and 38° respectively?

SUGGESTION. Use Fig. 415 and find AC.

$$\text{Then } \frac{555}{x + AC} = \tan 20°.$$

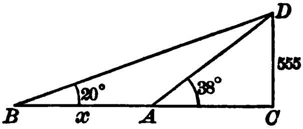


FIG. 415

10. A man standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank is 66°; when he goes back 45 ft. from the edge of the river, the angle of elevation is 42°. Find the height of the tree and the width of the river.

11. From the top of a hill two consecutive milestones on a straight horizontal road at the base of the hill are observed, and the angles of depression are found to be 13° and 8° respectively. Find the height of the hill.

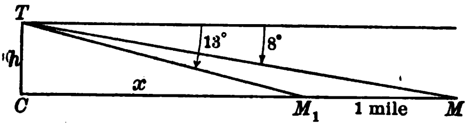


FIG. 416

HINT. Use Fig. 416 and construct $TC \perp CM$.

Let $CM_1 = x$.

Then $\frac{x}{h} = \tan 77°$, (1) Why?

and $\frac{x+1}{h} = \tan 82°$. (2) Why?

Subtracting (1) from (2), $\frac{1}{h} = \tan 82° - \tan 77°$.

Consult the table on page 315, substitute, and solve for h.

12. A ship has sailed due southwest a distance of 3.05 mi. How far is the ship south of the starting point? How far is it west of the starting point?

13. From the top of a mountain 3260 ft. above sea level the angle of depression of a boat on the ocean is 41°. How far is the boat from the top of the mountain?

14. The piece of artillery shown in Fig. 417 is set in position 6.9 mi. back from the bank of the river AB. If the maximum range of the gun is 9.9 mi., what distance along the river is within the area commanded by the piece?

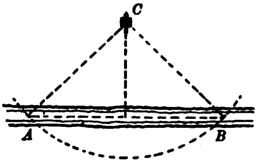


FIG. 417

NOTE. The river is assumed to be straight.

CHAPTER IX

LAWS OF FRACTIONS; FRACTIONAL EQUATIONS

412. Extension of the fundamental laws to fractions. The fundamental laws of addition, subtraction, multiplication, and division are extended to apply to fractions. In certain kinds of work it is important to possess skill in handling fractions. A very brief study of fractions is here made for two purposes: (1) to give greater meaning to the common processes of arithmetic and (2) to give the student power to put a formula into the form most convenient for practical use.

413. Reduction of fractions. We shall now consider a few simple exercises which review some of the well-known principles of arithmetic.

EXERCISES

Supply the missing numerators in each of the following equations:

1. $\frac{1}{2}=\frac{?}{4}$; $\frac{1}{2}=\frac{?}{6}$; $\frac{1}{2}=\frac{?}{10}$; $\frac{1}{2}=\frac{?}{12}$.

2. $\frac{8}{10}=\frac{?}{5}$; $\frac{6}{12}=\frac{?}{2}$; $\frac{4}{6}=\frac{?}{3}$; $\frac{6}{12}=\frac{?}{6}$.

3. How many twelfths are there in $\frac{18}{24}$? How many eighths? How many fourths? How many halves?

4. How many sixths are there in $\frac{1}{3}$? How many ninths? How many twelfths?

5. Show that $\frac{64}{72}=\frac{32}{36}=\frac{8}{9}$.

6. Show that if the principle in Ex. 5 is extended to algebraic numbers, then $\frac{b}{bx}=\frac{1}{x}$.

7. Show that $\frac{a}{e}=\frac{ab}{eb}$.

8. State the law which you use in changing a fraction to lower terms; to higher terms.

414. Fundamental principles. The preceding exercises illustrate the following familiar principles of arithmetic:

1. *Dividing the numerator and denominator of a fraction by the same number does not change the value of the fraction.*

This principle is used to change a fraction to lower terms.

2. *Multiplying the numerator and denominator of a fraction by the same number does not change the value of the fraction.*

This principle is used to change a fraction to higher terms.

3. *Reducing a fraction to lowest terms means finding the ratio between two numbers.*

415. ***Problem.*** *Reduce* $\frac{135\,x^3}{840\,x^2}$ *to lowest terms.*

First solution. Dividing first by x^2, then by 5, and then by 3, we have

$$\frac{135\,x^3}{840\,x^2}=\frac{135\,x}{840}=\frac{27\,x}{168}=\frac{9\,x}{56}.$$

Second solution. Dividing both the numerator and denominator at once by the greatest common divisor (in this case $15\,x^2$), we have

$$\frac{135\,x^3}{840\,x^2}=\frac{9\,x}{56}.$$

The above solutions show that we may divide out one factor at a time or we may divide both numerator and denominator at once by the product of all the common factors. It is usually easier to see one of the common factors than it is to see the greatest common divisor (G.C.D.). For example, it is always easy to tell whether or not 2, 5, or 10 is a common divisor. Why? The successive quotients are then easily simplified in turn.

EXERCISES

Reduce the following fractions to lowest terms:

1. $\frac{8}{20}$.
2. $\frac{21}{42}$.
3. $\frac{ax}{ay}$.
4. $\frac{ab}{ac^2}$.
5. $\frac{rs^2y}{rs^2m}$.
6. $\frac{x(x+y)}{y(x+y)}$.
7. $-\frac{a^2b^3}{a^3b^2}$.
8. $\frac{x^8}{x^3}$.
9. $\frac{x^5}{x^9}$.
10. $\frac{x^{n+8}}{x^n}$.
11. $\frac{x^n}{x^{n+1}}$.
12. $\frac{x^{n+10}}{x^{n+8}}$.
13. $\frac{x^{n+4}y^n}{x^n y^{n+5}}$.
14. $\frac{cx+cy}{ax+ay}$.
15. $\frac{axy-bxy}{ax^2y^2-bx^2y^2}$.
16. $\frac{(a+b)^4(c-d)^8}{(a+b)^8(c-d)^8}$.
17. $\frac{a^8b^4c^6(a+b)^5(a^2+b^2)^7}{a^2b^8c^5(a+b)^4(a^2+b^2)^8}$.

Historical Note. A sacred papyrus, written by an Egyptian priest named Ahmes (perhaps 2000 B.C.) and entitled "Directions for Knowing All Dark Things," furnishes us with the earliest work in the reduction of fractions. In one case he indicates his method as follows:

$$\tfrac{2}{3} = \tfrac{1}{2} + \tfrac{1}{6},$$

and therefore

$$\tfrac{2}{3} \text{ of } \tfrac{1}{5} = \tfrac{1}{2} + \tfrac{1}{6} \text{ of } \tfrac{1}{5};$$

that is,

$$\tfrac{1}{10} + \tfrac{1}{30}.$$

The practice of reducing fractions to the sum of several fractions, each of which had 1 for a numerator, remained with the Greeks until the sixth century of our own era,—the sole exceptions to this rule being the fractions $\frac{2}{3}$ and $\frac{3}{4}$. The reason why so much attention was given to fractions, Ball thinks, was the fact that in early times their treatment presented great difficulty.[1]

---

[1] See Ball, "A Short History of Mathematics."

The Romans usually kept the denominator constant and equal to 12, expressing the fraction (approximately) as so many twelfths; that is, they expressed all the fractions of a sum in terms of fractions which had 12 as a common denominator. Cajori says that they used 12 as the number of weights and measures and that they divided their coins into 12 *unciae* and the foot into 12 inches.

The Babylonians adopted a similar plan in connection with their study of astronomy, except that they used 60 as the constant denominator. From them through the Greeks the modern division of a degree into 60 equal parts is derived. Thus, in one way or another they all attempted to evade the difficulty of having to consider changes in both numerator and denominator.

The Hindus are said to have been the first to reduce fractions to a common denominator, although Euclid deduced a rule for finding the least common multiple. The Hindus wrote the numerator over the denominator without a line of separation. According to Ball the Arabs introduced the line to separate the numerator from the denominator, and thus the line came to be used as a symbol of division.

416. Combining fractions with the same denominators. We shall now review the method of combining fractions which have the same denominator.

EXERCISES

1. Add the following fractions:

(a) $\frac{2}{6}+\frac{5}{6}$. (c) $\frac{a}{c}+\frac{b}{c}+\frac{d}{c}$.

(b) $\frac{1}{2}+\frac{4}{2}+\frac{2}{2}$. (d) $\frac{m}{x}+\frac{n}{x}+\frac{e}{x}$.

2. Make a rule for adding fractions which have the same denominator.

3. Subtract as indicated in the following expressions:

(a) $\frac{6}{7}-\frac{3}{7}$. (b) $\frac{8}{5}-\frac{3}{5}-\frac{2}{5}$. (c) $\frac{m}{x}-\frac{n}{x}-\frac{c}{x}$.

4. Make a rule for subtracting fractions which have the same denominator.

5. Write each of the following expressions as one fraction:

(a) $\frac{17}{5}+\frac{3}{5}-\frac{9}{5}-\frac{4}{5}+\frac{3}{5}$. (b) $\frac{2a}{y}-\frac{3b}{y}+\frac{2c}{y}$.

6. Make a rule for combining a series of fractions which have the same denominator and are connected by plus and minus signs.

7. Combine each of the following expressions into a single fraction:

(a) $\frac{12}{21}+\frac{8}{21}+\frac{15}{21}$. (e) $\frac{a}{y}-\frac{b}{y}-\frac{c}{y}$.

(b) $\frac{x}{3}-\frac{y}{3}$. (f) $\frac{a+b}{3}+\frac{a-b}{3}$.

(c) $\frac{12}{23}+\frac{8}{23}-\frac{15}{23}$. (g) $\frac{3a-b}{z-y}-\frac{2a-3b}{z-y}$.

(d) $\frac{x}{b}-\frac{y}{b}$. (h) $\frac{a-2b}{x+3y}-\frac{7a+5b}{x+3y}$.

417. Adding fractions with different denominators. In adding fractions with different denominators we should remember that only like fractions can be added. Hence the problem of adding two fractions with different denominators must be changed to a problem of adding fractions with the same denominators. For example, if we are asked to add $\frac{1}{2}$ and $\frac{1}{3}$, we must write the problem as follows: What is the sum of $\frac{3}{6}+\frac{2}{6}$? The problem is then easy, since it is only necessary to add an integral (whole) number

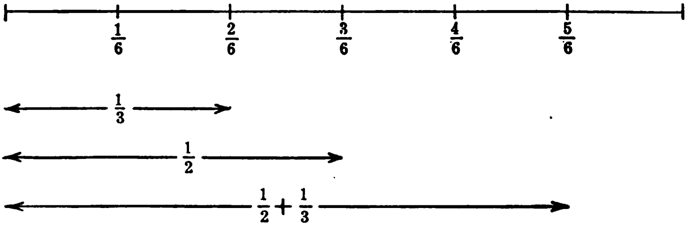


Fig. 418

of sixths to another integral number of sixths, and we are already familiar with the method of adding integers. Thus, in the above problem we see that $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$, a fact which is graphically illustrated in Fig. 418.

418. Adding unit fractions. Fractions like $\frac{1}{2}$ and $\frac{1}{3}$, where the numerator is 1, are called *unit fractions.*

EXERCISES

1. Add the following:

(a) $\frac{1}{3}+\frac{1}{5}$. (b) $\frac{1}{3}+\frac{1}{4}$. (c) $\frac{1}{5}+\frac{1}{4}$. (d) $\frac{1}{a}+\frac{1}{b}$ (e) $\frac{1}{x}+\frac{1}{y}$.

2. What is a short cut for adding unit fractions?

3. Using the rule found in Ex. 2, find the following sums:

(a) $\frac{1}{6}+\frac{1}{5}$. (e) $\frac{1}{3}+\frac{1}{y}$. (i) $\frac{1}{a+b}+\frac{1}{a}$.

(b) $\frac{1}{7}+\frac{1}{6}$. (f) $\frac{1}{7}+\frac{1}{b}$. (j) $\frac{1}{a+b}+\frac{1}{a-b}$.

(c) $\frac{1}{2}+\frac{1}{5}$. (g) $\frac{1}{x}+\frac{1}{z}$. (k) $\frac{1}{2x+3b}+\frac{1}{3x-2b}$.

(d) $\frac{1}{7}+\frac{1}{9}$. (h) $\frac{1}{m}+\frac{1}{n}$. (l) $\frac{1}{a+b}+\frac{1}{c+d}$.

419. Subtraction of unit fractions. A few simple examples will enable the student to develop a short method for use in subtracting unit fractions.

EXERCISES

1. Perform the indicated subtractions:

(a) $\frac{1}{2}-\frac{1}{3}$. (b) $\frac{1}{3}-\frac{1}{4}$. (c) $\frac{1}{a}-\frac{1}{b}$. (d) $\frac{1}{x+y}-\frac{1}{x+z}$.

2. Write out the work of each of the parts of Ex. 1 in the following form:

$$\frac{1}{4}-\frac{1}{6}=\frac{6}{4\cdot 6}-\frac{4}{4\cdot 6}=\frac{6-4}{4\cdot 6}=\frac{2}{24}=\frac{1}{12}.$$

3. From Ex. 2 state a short method for finding the difference when subtracting one unit fraction from another.

4. Apply the rule of Ex. 3 to the following problems:

(a) $\frac{1}{3} - \frac{1}{7}$. (d) $\frac{1}{3} - \frac{1}{z}$. (g) $\frac{1}{x} - \frac{1}{y}$.

(b) $\frac{1}{4} - \frac{1}{8}$. (e) $\frac{1}{a} - \frac{1}{c}$. (h) $\frac{1}{x} - \frac{1}{x+y}$.

(c) $\frac{1}{7} - \frac{1}{a}$. (f) $\frac{1}{b} - \frac{1}{a}$. (i) $\frac{1}{x+y} - \frac{1}{x-y}$.

5. Show that the rule for the addition and subtraction of unit fractions may be written in the form $\frac{1}{x} \pm \frac{1}{y} = \frac{y \pm x}{xy}$.

420. General law for the addition and subtraction of fractions. A simple illustration will show that any two fractions may be added as easily as two unit fractions.

For example, add $\frac{2}{3}$ and $\frac{4}{5}$.

Solution. Before we can add $\frac{2}{3}$ and $\frac{4}{5}$ we must change each fraction to another fraction with a denominator which is the least common denominator (L.C.D.) of the two given fractions. In this case the L.C.D. is 15.

Then $\frac{2}{3} = \frac{10}{15}$ and $\frac{4}{5} = \frac{12}{15}$.

Hence $\frac{10}{15} + \frac{12}{15} = \frac{22}{15}$.

The work may be shortened if it is arranged as shown below, the arrows indicating the steps in the process.

$$\frac{2}{3} + \frac{4}{5} = \frac{2 \cdot 5}{3 \cdot 5} + \frac{3 \cdot 4}{3 \cdot 5}$$

EXERCISES

1. Add $\frac{4}{5}$ and $\frac{8}{9}$; $\frac{a}{b}$ and $\frac{c}{d}$.

2. Add $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$, where n_1 and n_2 represent the numerators and d_1 and d_2 the denominators respectively.

3. Express the result of Ex. 2 as a general law for the algebraic addition of fractions.

4. Show that the law of Ex. 3 may be written in the form

$$\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_1d_2+n_2d_1}{d_1d_2}.$$

5. Subtract $\frac{3}{7}$ from $\frac{5}{9}$; $\frac{5}{6}$ from $\frac{6}{7}$.

6. Subtract $\frac{n_2}{d_2}$ from $\frac{n_1}{d_1}$.

7. From Ex. 6 state a short method for finding the difference of two fractions.

8. Perform the following operations:

(a) $\frac{5}{7}+\frac{3}{13}$.

(b) $\frac{5}{11}-\frac{3}{14}$.

(c) $\frac{x}{5}+\frac{y}{6}$.

(d) $\frac{y}{a}-\frac{x}{b}$.

(e) $\frac{8}{x}+\frac{5}{z}$.

(f) $\frac{m+2}{3}+\frac{m+3}{4}$.

(g) $\frac{a+b}{5}-\frac{c+b}{6}$.

(h) $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

(i) $\frac{2a-3b}{a-2b}-\frac{a+2b}{2a+3b}$.

(j) $\frac{3x+4y}{3x-4y}+\frac{3x-4y}{3x+4y}$.

421. Fractions with one or more common factors in the denominator. The preceding short methods are of no particular value when the denominators of the fractions to be added have a common factor. In such cases before the fractions are added they should be reduced to fractions which have for their denominators the least common multiple (L.C.M.) of the given denominators. For example, consider the following cases:

$$\tfrac{2}{3}+\tfrac{5}{6}+\tfrac{3}{4}=\tfrac{8}{12}+\tfrac{10}{12}+\tfrac{9}{12}=\tfrac{27}{12};$$

$$\frac{a}{x^2}+\frac{b}{xz}-\frac{c}{z^2}=\frac{az^2}{x^2z^2}+\frac{bxz}{x^2z^2}-\frac{cx^2}{x^2z^2}=\frac{az^2+bxz-cx^2}{x^2z^2}.$$

EXERCISES

Using the preceding method, add and subtract as indicated:

1. $\frac{m}{2}+\frac{n}{5}+\frac{s}{15}$.

2. $\frac{3a}{4}+\frac{4b}{6}+\frac{5c}{3}$.

3. $\frac{5b}{24}-\frac{3b}{20}$.

4. $\frac{m}{dx}+\frac{n}{dy}$.

5. $\frac{l}{t}-\frac{b}{ta}$.

6. $\frac{t}{24ab}-\frac{m}{6b}$.

7. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$.

8. $\frac{x}{x-1}-\frac{bx}{x(x-1)}$.

9. $\frac{4}{y-1}-\frac{1}{3(y-1)}$.

10. $\frac{1}{p+q}+\frac{1}{p-q}$.

422. Adding integers and fractions. The student often fails to see that the preceding laws apply when an integer is added to a fraction. Since any integer (say 4) may be thought of as a fraction, $4+\frac{2}{3}$ may be written $\frac{4}{1}+\frac{2}{3}=\frac{12}{3}+\frac{2}{3}=\frac{14}{3}$. A similar statement is true for subtraction.

EXERCISES

Add or subtract as indicated:

1. $1+\frac{9}{13}$.

2. $6+\frac{5}{9}$.

3. $8-\frac{7}{12}$.

4. $\frac{c}{d}+1$.

5. $\frac{a}{b}-1$.

6. $1+\frac{x}{y}$.

423. Addition law of proportion. The work of the preceding article may be used to establish the addition and subtraction laws of proportion. If we have the given proportion

$$\frac{a}{b}=\frac{c}{d},$$

then

$$\frac{a}{b}+1=\frac{c}{d}+1, \qquad \text{Why?}$$

or

$$\frac{a+b}{b}=\frac{c+d}{d},$$

which is the addition law of proportion.

EXERCISES

Apply the preceding law of addition to the following proportions:

1. $\frac{20}{15}=\frac{4}{3}$; $\frac{25}{40}=\frac{5}{8}$.
2. $\frac{x+2}{x-2}=\frac{y+4}{y-4}$.

424. Subtraction law of proportion. If we have the given proportion

$$\frac{a}{b}=\frac{c}{d},$$

then

$$\frac{a}{b}-1=\frac{c}{d}-1, \qquad \text{Why?}$$

or

$$\frac{a-b}{b}=\frac{c-d}{d},$$

which is the subtraction law of proportion.

EXERCISES

Apply the preceding subtraction law to the following proportions:

1. $\frac{8}{3}=\frac{40}{15}$; $\frac{7}{5}=\frac{35}{25}$.
2. $\frac{x+5}{x-5}=\frac{y-6}{y+6}$.

425. Practice in the addition and subtraction of fractions. The following exercises furnish practice in the addition and subtraction of fractions.

EXERCISES

Add and subtract as indicated:

1. $\frac{3a}{2}-\frac{7a}{4}+\frac{13a}{12}$.
2. $\frac{ab}{6}+\frac{bc}{18}-\frac{ac}{12}$.
3. $\frac{3x-2}{6}+\frac{6x+4}{9}$.
4. $\frac{4x-9}{18}-\frac{3x-6}{12}$.
5. $\frac{5x+2}{3}+\frac{4y+3}{6}-\frac{7z-5}{9}$.
6. $\frac{3b+4}{3}-\frac{4b-5}{4}+\frac{5b+3}{6}$.

7. $\frac{2}{x}-\frac{7}{2x}$.

8. $\frac{2}{3b}+\frac{5}{4a}$.

9. $\frac{2}{x^2}-\frac{2}{xy}+\frac{3}{y^2}$.

10. $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}$.

11. $\frac{a}{y}+b\left(b-\frac{by}{b}\right)$.

12. $\frac{x}{z}-a$.

13. $5x+\frac{2x}{4}$.

14. $\frac{a+b}{x}+\frac{a-b}{4x}$.

15. $\frac{a-3b}{3x}-\frac{2a-5b}{4x}$.

16. $\frac{1}{x^3}-\frac{1}{x^2}+\frac{1}{x}$.

17. $\frac{5a-4b+3c}{a}+\frac{2a+3b-4c}{b}$.

18. $\frac{7x+2y-3c}{x}-\frac{2x+4y-3c}{y}$.

19. $\frac{x+y}{x}-\frac{x}{x-y}$.

20. $\frac{(x+y)^2}{4xy}-1$.

21. $\frac{a}{a^2-1}+\frac{a+3}{a-1}-\frac{a-3}{a+1}$.

22. $\frac{a}{a^2-1}+\frac{a-2}{a^2+2a+1}-\frac{a+1}{a+1}$.

23. $\frac{2x}{x^2-y^2}+\frac{3x}{x^2-2y^2+y^2}+\frac{4x}{x-y}$.

24. $\frac{a+b-c}{(a+c)(b+c)}+\frac{a-b+c}{(a+b)(b+c)}-\frac{x+y+z}{(a+b)(a+c)}$.

426. Applying the laws of fractions to formulas. The laws which we have discussed in this chapter are useful in helping us to put certain formulas in a form more convenient for practical use, as is shown in the exercises on the following page.

EXERCISES

Express each of the right-hand members of the following formulas as a single fraction (leave the left-hand member as given):

1. $\frac{1}{R}=\frac{1}{P}+\frac{1}{Q}.$

Solution. $\frac{1}{R}=\frac{Q}{PQ}+\frac{P}{PQ}=\frac{Q+P}{PQ}.$

2. $\frac{1}{K}=\frac{1}{P}-\frac{1}{Q}.$

3. $\frac{1}{M}=\frac{1}{P}+\frac{2}{q}.$

4. $\frac{1}{N}=\frac{3}{R}-\frac{5}{t}.$

5. $\frac{1}{S}=1-\frac{P}{q}.$

6. $\frac{1}{x}=\frac{P}{q}+r.$

7. $\frac{1}{V}=\frac{P}{q}+\frac{1}{r}.$

8. $\frac{1}{W}=\frac{2P}{q}-\frac{3}{r}.$

9. $\frac{1}{y}=\frac{3P}{r}+\frac{2}{q}.$

10. $\frac{1}{R}=\frac{2}{P}+\frac{3}{Q}.$

427. Multiplication of fractions. The method of procedure to be followed in the multiplication of fractions may be illustrated geometrically as follows:

For example, a man willed three fourths of his estate to his two sons, the elder son receiving two thirds of this amount. What part of the entire estate did the elder son receive?

Solution. Let the rectangle (Fig. 419) represent the entire estate. Divide the rectangle into fourths by the lines MN, PQ, and RS, and into thirds by the lines EF and GH.

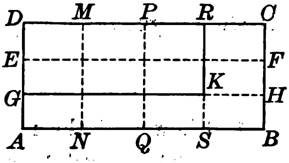


FIG. 419

The elder son received $\frac{2}{3}$ of $\frac{3}{4}$ of the entire estate; therefore his part is represented by the rectangle $GKRD$. This is easily seen to be $\frac{6}{12}$, or $\frac{1}{2}$, of the entire rectangle.

EXERCISES

Illustrate each of the following products by means of a freehand drawing:

1. $\frac{2}{3}$ of $\frac{3}{5}$.
2. $\frac{2}{5}$ of $\frac{3}{4}$.
3. $\frac{3}{4}$ of $\frac{2}{3}$.
4. $\frac{3}{4}$ of $\frac{5}{6}$.

428. Law for multiplying one fraction by another. In general, the product of two fractions may be interpreted in the same way as were those in the preceding exercises, for if the two factors are the integers x and y $\left(\text{which may be written } \frac{x}{1} \text{ and } \frac{y}{1}\right)$, then the product (see Fig. 420) is xy $\left(\text{which may be written } \frac{xy}{1}\right)$. But if one of the factors is $\frac{x}{2}$, then the product is $\frac{xy}{2}$ (see Fig. 420).

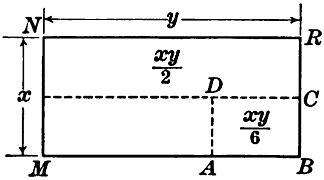


FIG. 420

Furthermore, if the other factor is now reduced to one third its original size $\left(\text{the factor becomes } \frac{y}{3}\right)$, then the product undergoes a similar reduction (see Fig. 420); that is, $\frac{x}{2}\cdot\frac{y}{3}=\frac{xy}{6}$.

Thus the product of $\frac{x}{2}$ and $\frac{y}{3}$ is $\frac{xy}{6}$, and this product is shown geometrically by the rectangle $ABCD$ (Fig. 420).

EXERCISES

1. Show that the rectangle $MBRN$ in Fig. 421 is six times as large as the rectangle $ABCD$.

2. By means of a geometric figure show the product of the following factors:

(a) $\frac{a}{4}$ and $\frac{b}{3}$. (b) 6 and $\frac{b}{3}$.

3. In what way do the figures in Ex. 2 show that *the product of two fractions is the fraction obtained by multiplying together the respective numerators and denominators of the fractions?*

429. Extension of the multiplication law. Obviously the law for multiplying two fractions can be extended to give us the product of more than two fractions. For example, it is seen at once that $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{6}{8} = \frac{3}{8}$ if we divide out the factors that are common to the numerators and the denominators. This division by common factors reduces the product to lowest terms, and the work is usually much simpler if this is done before the actual multiplication of the fractions.

EXERCISES

Give the products of the following fractions:

1. $\frac{2}{3} \cdot \frac{4}{7}$; $\frac{1}{3} \cdot \frac{2}{3}$; $\frac{3}{4} \cdot \frac{5}{6} \cdot \frac{2}{5}$.

2. $\frac{a}{5} \cdot \frac{b}{3}$; $\frac{a}{b} \cdot \frac{c}{d}$; $\frac{a}{b} \cdot \frac{x}{y} \cdot \frac{b}{a}$.

3. $\frac{n_1}{d_1} \cdot \frac{n_2}{d_2}$; $\frac{x^2}{y^2} \cdot \frac{x^3}{y^3}$; $\frac{a^2}{x^3} \cdot \frac{x^2}{a^3} \cdot \frac{b}{a}$.

4. $\frac{1}{x} \cdot xy$; $\frac{1}{x^2} \cdot x^5$; $\frac{x^2}{y^2} \cdot \frac{y^7}{x^3} \cdot xy$.

5. $\frac{a}{x} \cdot \frac{x}{b}$; $\frac{c}{y} \cdot \frac{z}{c}$; $\frac{ab}{x} \cdot \frac{cd}{y} \cdot \frac{yx}{acd}$.

6. $\frac{5abc}{2mnr} \cdot \frac{4mn}{10a^2bc^3} \cdot 2c^2$.

7. $\frac{2xy}{3zw} \cdot \frac{4tw}{10my}$.

8. $\frac{4mnt}{3rsx} \cdot \frac{5xz}{8mw} \cdot \frac{7rw}{9tz}$.

9. $\frac{3a^2y}{5c^2z} \cdot \frac{9cz^3}{6yb^3} \cdot \frac{4by^3}{3c^2z^2}$.

10. $\frac{x+y}{x^2+y^2} \cdot \frac{x^2-y^2}{x-y}$.

11. $\frac{12a^2x}{3b^2y} \cdot \frac{6by^2}{7ax^2} \cdot \frac{7b}{14a}$.

12. $\frac{a+b}{2} \cdot \frac{5c}{3x+3z}$.

13. $\frac{a^2b^2}{3x^2y^3} \cdot \frac{6x^4y^2}{5a^4b^2}$.

14. $\frac{x^2}{x-y} \cdot \frac{y^2}{x+y}$.

15. $\frac{a+b}{a-b} \cdot \frac{a^2-b^2}{a^2+b^2}$.

16. $\frac{a^2-9}{a+3} \cdot \frac{a-3}{a^2-6x+9}$.

17. $\frac{a}{a+b} \cdot \frac{b}{a-b}$.

18. $\frac{5a-5b}{3m^2n} \cdot \frac{21mn^3}{30a-30b}$.

19. $\frac{16(x-y)}{5x^2y^2} \cdot \frac{15x^2y^3}{4(x-y)}$.

20. $\frac{x^2+xy}{m^2-n^2} \cdot \frac{m^2+n^2}{x^2-xy}$.

21. $\frac{a^2-ab}{x^2-xy}\cdot\frac{x^2+xy}{a^2+ab}$.

22. $\frac{3a-3b}{a^2+2ab+b^2}\cdot\frac{7a+7b}{2a^2-2b^2}$.

23. $\frac{7xyz-28}{3my-3ny}\cdot\frac{21my-21ny}{3xyz+12x}$.

24. $\frac{3a-3b}{a^2-ab+b^2}\cdot\frac{a^2+ab+b^2}{2a+2b}$.

25. $\frac{a+1}{a-1}\cdot\frac{a+2}{a-2}\cdot\frac{a+3}{a-3}$.

430. Division of fractions. We can understand best how to divide one fraction by another if we illustrate a concrete problem by a geometric drawing.

For example, a farmer has two plots of ground of equal size, parts of which he plans to give to his two sons for gardens. He gives two thirds of one plot to one son and three fourths of the other plot to the second son. The second son's garden is what fractional part of the first son's garden?

Solution. Let the rectangle $ABCD$ (Fig. 421) represent the size of each original plot of ground. Divide the rectangle into thirds and fourths as shown. The rectangle $ABFE$ (two thirds of the original plot) contains eight of the small rectangles. The rectangle $ASRD$ (three fourths of the original plot) contains nine of the small rectangles.

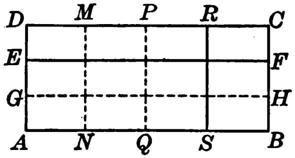


FIG. 421

Since the small rectangles are the same size, the quotient obtained by dividing the number of rectangles (9) in the second son's garden by the number of rectangles (8) in the first son's garden is $\frac{9}{8}$.

Hence $\frac{3}{4}\div\frac{2}{3}=\frac{9}{8}$, and the second son's garden is $\frac{9}{8}$ of the first son's garden.

The result may be checked by multiplying $\frac{2}{3}$ by $\frac{9}{8}$, and we have $\frac{2}{3}\cdot\frac{9}{8}=\frac{18}{24}=\frac{3}{4}$.

EXERCISES

1. Show that the check in the above problem agrees with the following definition: *Division is the process of finding one of two numbers, called the quotient, which when multiplied by the other, called the divisor, gives a third number, called the dividend.*

2. Illustrate each of the following divisions by means of a freehand drawing, find each quotient, and check each result:

(a) $\frac{2}{5} \div \frac{3}{4}$. (b) $\frac{3}{5} \div \frac{2}{3}$. (c) $\frac{2}{5} \div \frac{5}{6}$. (d) $\frac{3}{4} \div \frac{2}{7}$.

431. Law for dividing one fraction by another. The preceding exercises illustrate the following law: *To divide one fraction by another multiply the first fraction (the dividend) by the inverted divisor*; that is, by the *reciprocal* of the divisor.

EXERCISES

Divide as indicated:

1. $a \div \frac{a}{b};\ a \div \frac{b}{a}$.

2. $5 \div \frac{x}{y};\ 12\,a^3 \div \frac{4\,a^2}{7\,y}$.

3. $36\,x^3y^3z^3 \div \frac{18\,x^3y^3z}{ab}$.

4. $\frac{32\,x^3y^2}{27\,a^3b^5} \div \frac{8\,x^3y}{9\,a^2b^4}$.

5. $\frac{20\,a^2b^3}{21\,x^4y^5} \div \frac{4\,ab^2y}{3\,x^2y^2}$.

6. $\frac{12(x-2)^2}{35(m^2-n^2)} \div \frac{15(x-2)}{14(m-n)^2}$.

7. $\frac{a+b}{b} \div \frac{a-b}{b};\ \frac{c+d}{d} \div \frac{c-d}{d}$.

8. $\frac{x^2+9}{x^2-9} \div \frac{(x+3)^2}{x-3}$.

9. $\frac{a^2+2\,ab+b^2}{x^2-y^2} \div \frac{a^2-b^2}{(x+y)^2}$.

432. Complex fractions. If the members of the equation $\frac{a+b}{b}=\frac{c+d}{d}$ are divided by the corresponding members of the equation $\frac{a-b}{b}=\frac{c-d}{d}$, we have

$$\frac{\frac{a+b}{b}}{\frac{a-b}{b}}=\frac{\frac{c+d}{d}}{\frac{c-d}{d}};$$

whence
$$\frac{a+b}{a-b}=\frac{c+d}{c-d},$$

which expresses the law stated in Art. 348, Ex. 2.

Such a fraction as $\frac{\frac{a+b}{b}}{\frac{c-d}{d}}$ is called a *complex fraction.*

Hence *a complex fraction is a fraction in which either the numerator, the denominator, or both contain fractions.*

EXERCISES

1. Prove the preceding law; that is, prove that if

$$\frac{a}{b}=\frac{c}{d}, \quad \text{then} \quad \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

2. Using the law in Ex. 1, solve for y in the following proportions:

(a) $\frac{5-y}{y+3}=\frac{2-y}{4+y}$.

(b) $\frac{y-4}{y-12}=\frac{y-8}{y-13}$.

(c) $\frac{y+3}{y+2}=\frac{y+1}{y+3}$.

(d) $\frac{2y+3}{5-2y}=\frac{3y+4}{5-3y}$.

433. Geometric application of the laws of Arts. 347 and 348. In Fig. 422 the line ED is parallel to BC. In Arts. 353 and 354 we showed that

$$\frac{AE}{EB}=\frac{AD}{DC}$$

and that $$\frac{AB}{AE}=\frac{AC}{AD}.$$

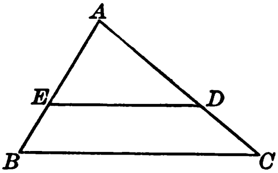


FIG. 422

If we take the equation

$$\frac{AE}{EB}=\frac{AD}{DC}$$

and use the law of Art. 347, we have

$$\frac{AE+EB}{EB}=\frac{AD+DC}{DC},$$

or $$\frac{AB}{EB}=\frac{AC}{DC}.$$

The preceding equation expresses the following well-known theorem: *If a line is drawn parallel to the base of a triangle, it divides the other two sides proportionally.*

Similarly, by using the law of Art. 348, Ex. 1, and the equation

$$\frac{AB}{AE}=\frac{AC}{AD},$$

we have $$\frac{AB-AE}{AE}=\frac{AC-AD}{AD},$$

or $$\frac{EB}{AE}=\frac{DC}{AD},$$

which is another statement of the theorem above.

The student should be clear as to the two methods for proving this theorem.

APPLIED PROBLEMS

1. Two rectangular rooms are to be made into one room. The area of one of the rooms is A_1 (Fig. 423), that of the other is A_2; the width of the first room is w_1, that of the second is w_2. Find the length of the new room.

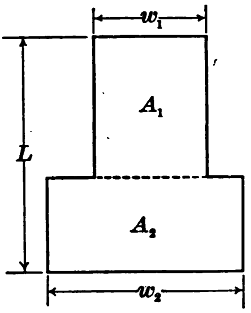


Fig. 423

2. What is the length of the new room in Ex. 1 if the first room is twice as large (both dimensions) as the second?

3. A stenographer can copy a certain manuscript on the typewriter in 5 hr. Her assistant can do the same work in 7 hr. Find what part of the work they could do together in 1 hr.

4. If the stenographer in Ex. 3 could do the work in f hours and the assistant in t hours, what part could they do together in 1 hr.?

5. Working together, how long would it take the stenographers in Ex. 3 to do the work?

6. In Ex. 3 write a formula for the part of the work that can be done together in 1 hr. by two stenographers who work at the same rate as the first one, and by three stenographers who work at the same rate as her assistant.

7. In a guessing game the leader said, "If you will add 10 yr. to your age, divide the sum by your age, add 6 to the quotient, and tell me the result, I will tell you your age." How did he find the result?

8. A baseball team has played 40 games and has won 28 of them. How many games in succession must the team win in order to bring its average of games won up to 0.750?

9. A girl has worked 12 problems. If she should work 13 more and get 8 of them correct, her average would be 72 per cent. How many problems so far has she worked correctly?

10. The physicist uses the formula $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$, where F is the principal focal distance for a lens and f_1 and f_2 are the conjugate focal distances. How can you get a formula that is in the most convenient form for solving for F; that is, one in which F is an explicit function of f_1 and f_2 and not an implicit function as at present given?

11. What must be the length of the rectangle in Fig. 424 in order that its area may be $\frac{xy}{8}$ and its width $\frac{y}{2}$?

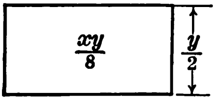


FIG. 424

12. A swimming pool is filled by two pipes, one of which delivers m gallons in 5 min. and the other n gallons in 3 min. Write a formula for calculating the number of gallons delivered in 1 min. when both pipes are running, expressing the result as one fraction.

13. Suppose that the second pipe in Ex. 12 can be used to drain the pool at the same rate. Write a formula for calculating the number of gallons per minute by which the amount of water in the pool is increased when both pipes are open, expressing the result as a single fraction.

14. Two cyclists ride together around a circular track, one along the outside edge, where the radius of the circle is R, and the other along the inside edge, where the radius is r. One revolution of the pedals carries the first bicycle a distance of 20 ft. and the second a distance of 25 ft. Write a formula which expresses the difference between the number of pedal revolutions made by the two cyclists in going around the track once; five times; n times.

15. When 10 marbles of a certain size are dropped into a bucket of water, the water rises m inches. In the same bucket it takes 15 marbles of another size to raise the water n inches. Write a formula which shall show how many times larger a marble of the first type is than one of the second type.

16. A certain automobile tire when fully inflated has a radius of R inches, but owing to a leakage of air the radius is reduced to r inches. Indicate how many more revolutions per mile are necessary because of the leakage. Express a formula for calculating the difference in the number of revolutions per mile.

434. Fundamental relations of trigonometric ratios; trigonometric identities. Many important relations between the sine, cosine, and tangent of any given acute angle can be established. These relations are often shown by very simple formulas. Since these relations are formulas and are true for all values of the letters involved, we shall call them *trigonometric identities.*

For example, if A is any acute angle, prove that

$$\sin^2 A + \cos^2 A \equiv 1.$$

NOTE. The symbol $\equiv$ is read "is identical with." The expression $\sin^2 A$ means $(\sin A)^2$, but the first form is the one generally used.

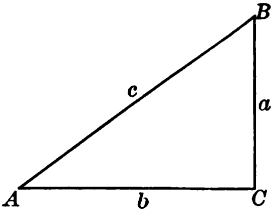


FIG. 425

Proof. In Fig. 425 we know that

$$\sin A = \frac{a}{c},$$

and $$\cos A = \frac{b}{c}.$$

$$\therefore \sin^2 A + \cos^2 A \equiv \frac{a^2}{c^2} + \frac{b^2}{c^2} \equiv \frac{a^2 + b^2}{c^2}.$$

We know that in any right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Then $$a^2 + b^2 \equiv c^2,$$

and $$\sin^2 A + \cos^2 A \equiv \frac{c^2}{c^2} \equiv 1.$$

$\therefore$ the truth of the identity is established.

Since the sine of an obtuse angle and the sine of its supplement are numerically equal, and the same is true for the cosines, the identity holds for obtuse angles as well as for acute angles.

EXERCISES

1. In Fig. 425 prove that $\sin^2 B + \cos^2 B \equiv 1$.

2. In Fig. 425 prove that $\tan A \equiv \dfrac{\sin A}{\cos A}$; $\tan B \equiv \dfrac{\sin B}{\cos B}$.

3. By means of the identity $\sin^2 A + \cos^2 A \equiv 1$ show that $\sin A \equiv \sqrt{1 - \cos^2 A}$ and that $\cos A \equiv \sqrt{1 - \sin^2 A}$.

NOTE. The ± sign is not used here because negative sines or cosines are meaningless for positive acute angles.

4. Using the formulas of Ex. 3, find $\sin A$ if $\cos A = \frac{3}{4}$; find $\cos A$ if $\sin A = \frac{3}{5}$.

5. Using the formulas of Ex. 2, find $\tan A$ if $\cos A = .75$ and $\sin A = .66$; find $\cos A$ if $\tan A = \frac{3}{4}$ and $\sin A = \frac{3}{5}$.

435. Method of proving trigonometric identities. Proving trigonometric identities will give the student further practice in the fundamental operations with fractions. There are several methods of proving identities, but we shall illustrate only three of the more common methods.

The first method consists in changing one member of the required identity to the form of the other by using another identity already known.

1. Prove that $\qquad 1 + \tan^2 A \equiv \dfrac{1}{\cos^2 A}$.

Proof. By Art. 434 we know that

$$\tan A \equiv \frac{\sin A}{\cos A}.$$

$$\therefore \tan^2 A \equiv \frac{\sin^2 A}{\cos^2 A}.$$

$$\therefore 1 + \tan^2 A \equiv 1 + \frac{\sin^2 A}{\cos^2 A} \equiv \frac{\cos^2 A + \sin^2 A}{\cos^2 A}.$$

And by Art. 434, $\quad \cos^2 A + \sin^2 A \equiv 1$.

$$\therefore \frac{\cos^2 A + \sin^2 A}{\cos^2 A} \equiv \frac{1}{\cos^2 A}.$$

$$\therefore 1 + \tan^2 A \equiv \frac{1}{\cos^2 A}.$$

The second method consists in changing one member to its simplest form and then changing the other member to the same form. This is usually done by changing each member to a function of the sine or cosine of the given angle.

2. Prove that $\dfrac{1}{\sin A} - \sin A \equiv \cos A \cdot \dfrac{1}{\tan A}$.

Proof. Taking only the left member, we have

$$\frac{1}{\sin A} - \sin A \equiv \frac{1-\sin^2 A}{\sin A} \equiv \frac{\cos^2 A}{\sin A}.$$

Now taking the right member, we have

$$\cos A \cdot \frac{1}{\tan A} \equiv \cos A \cdot \frac{1}{\dfrac{\sin A}{\cos A}} \equiv \frac{\cos^2 A}{\sin A}.$$

Then, because they are both equal to the same thing,

$$\frac{1}{\sin A} - \sin A \equiv \cos A \cdot \frac{1}{\tan A}.$$

A third method of proving identities is to use the method of analysis. This method consists of two parts — the analysis and the proof. In the analysis we assume that what we want to prove is true and thus discover how to prove the identity. In the proof we simply retrace the steps taken in the analysis.

3. Prove that $\dfrac{1}{\tan^2 A} \equiv \dfrac{1}{\sin^2 A} - 1.$

Analysis. We first assume that

$$\frac{1}{\tan^2 A} \equiv \frac{1}{\sin^2 A} - 1.$$

Then $$\frac{1}{\dfrac{\sin^2 A}{\cos^2 A}} \equiv \frac{1-\sin^2 A}{\sin^2 A},$$

or $$\frac{\cos^2 A}{\sin^2 A} \equiv \frac{\cos^2 A}{\sin^2 A}.$$

Proof. In proving the identity we start with

$$\frac{\cos^2 A}{\sin^2 A} \equiv \frac{\cos^2 A}{\sin^2 A}. \tag{1}$$

For the left member of (1) we can substitute its equal,

$$\frac{1}{\dfrac{\sin^2 A}{\cos^2 A}};$$

and in the right member, for $\cos^2 A$ we can substitute

$$1 - \sin^2 A.$$

Then

$$\frac{1}{\dfrac{\sin^2 A}{\cos^2 A}} \equiv \frac{1 - \sin^2 A}{\sin^2 A}. \tag{2}$$

But

$$\frac{\sin^2 A}{\cos^2 A} = \tan^2 A,$$

and hence the left member of (2) can be written

$$\frac{1}{\tan^2 A}.$$

The right member of (2) can be written

$$\frac{1}{\sin^2 A} - \frac{\sin^2 A}{\sin^2 A},$$

or

$$\frac{1}{\sin^2 A} - 1.$$

$$\therefore \frac{1}{\tan^2 A} \equiv \frac{1}{\sin^2 A} - 1.$$

EXERCISES

Prove the following identities:

1. $\cos A \equiv \dfrac{\sin A}{\tan A}$.
2. $\cos A \tan A \equiv \sin A$.
3. $\tan A \cdot \dfrac{\cos A}{\sin A} \equiv 1$.
4. $\cos^2 A - \sin^2 A \equiv 1 - 2\sin^2 A$.
5. $\cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1$.

6. $\tan A \cdot \cos A \cdot \dfrac{1}{\sin A} \equiv 1.$

7. $\dfrac{\cos A + \sin A}{1 + \tan A} \equiv \cos A.$

8. $\dfrac{1}{\sin^2 A} - 1 \equiv \dfrac{1}{\tan^2 A}.$

9. $\dfrac{\sin A}{\cos A \tan A} \equiv 1.$

10. $\dfrac{1}{\sin^2 A} - 1 \equiv \dfrac{1}{\tan^2 A}.$

11. $\dfrac{\sin A}{\cos A} + 1 \equiv 1 + \tan A.$

12. $\sqrt{1 - \sin^2 A} \equiv \sin A \cdot \dfrac{1}{\tan A}.$

13. $\sqrt{1 - \cos^2 A} \equiv \cos A \tan A.$

14. $\cos A \equiv \dfrac{1}{\cos A} - \sin A \tan A.$

436. Trigonometric equations. In algebra we often need to find the value of x which satisfies a given equation. Similarly, in trigonometry we often need to find the value of an angle x which satisfies an equation containing some trigonometric function or ratio. Such equations are called *trigonometric equations.*

The method of solving these equations is a little different from the method of solving algebraic equations.

For example, find the value of the angle x which satisfies the equation $\sin^2 x = \frac{1}{4}$.

Solution. Taking the square root of both sides, we have

$$\sin x = \tfrac{1}{2}.$$

But by Art. 403 we know that $\sin 30° = \frac{1}{2}$.

Therefore the equation $\sin^2 x = \frac{1}{4}$ is satisfied if 30° is substituted for x, and $x = 30°$ is a solution of the given equation.

EXERCISES

Solve the following trigonometric equations:

1. $\cos^2 x = \frac{1}{4}.$

2. $4 \sin^2 x = 3.$

Solution. Dividing both members of the equation by 4, we have

$$\sin^2 x = \tfrac{3}{4}.$$

François Viète, or Vieta, as he was commonly called, was born in France in 1540 and died in 1603. He was the most influential man of his time in advocating modern symbols in algebra. Although the symbols used a little later (1637) by Descartes were so much better that they superseded those of Vieta, the latter's influence in the general improvement of algebra was such that he ranked as one of the foremost makers of the science as we have it at present

Then $$\sin x = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

But by Art. 403 we know that $\sin 60° = \frac{\sqrt{3}}{2}$.

$\therefore x = 60°$ is a solution of the original equation.

3. $\sin^2 x = \frac{1}{2}$. **4.** $\cos x = \frac{1}{2\cos x}$. **5.** $\tan^2 x = 1$.

6. $\sin^2 x - \cos^2 x = \cos x$.

Solution. In the equation, for $\sin^2 x$ substitute
$$1 - \cos^2 x.$$

Then $$1 - \cos^2 x - \cos^2 x = \cos x,$$
or $$1 - 2\cos^2 x - \cos x = 0.$$

Rearranging the terms, we have
$$-2\cos^2 x - \cos x + 1 = 0,$$
or $$2\cos^2 x + \cos x - 1 = 0.$$

This is a quadratic equation in $\cos x$ which we can solve by factoring as follows:
$$(2\cos x - 1)(\cos x + 1) = 0.$$

The above product is 0, as shown, if
$$2\cos x - 1 = 0,$$
that is, if $$\cos x = \tfrac{1}{2};$$
or if $$\cos x + 1 = 0,$$
that is, if $$\cos x = -1.$$

Since we have considered only positive acute angles, $\cos x$ cannot be equal to -1.

Hence $$\cos x = \tfrac{1}{2}.$$
But $$\cos 60° = \tfrac{1}{2},$$
and hence $$x = 60°.$$

7. $2\tan x \equiv \frac{1}{\cos x}$. **11.** $\tan x + \frac{1}{\tan x} = 2$.

8. $3\tan x \cos x = \frac{3}{2}$. **12.** $\sin x = \cos^2 x - \sin^2 x$.

9. $2\sin^2 x = 3\cos x$. **13.** $\tan x + \tan^2 x = 3\tan x - 1$.

10. $\tan^2 x + 3 = 4\tan x$. **14.** $\frac{5}{\cos^2 x} = 11 - \tan^2 x$.

CHAPTER X

AREAS

437. Measuring areas. If we determine the amount of area inclosed within a polygon, as in the $\triangle ABC$ in Fig. 426, we are *measuring the area* of the polygon. As in measuring length, the process is one of comparison. We compare the area of the given polygon with some standardized (defined and accepted) unit of area and determine how many units of area are contained in the polygon; that is, we determine the ratio between the area of the given polygon and a standard unit of area.

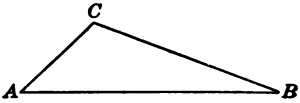


FIG. 426

438. Unit of area. The unit of area is a square each of whose sides is a standard unit of length. Such a unit involves length and width and is therefore a numerical measure. Thus, we may measure area and express the result in square inches, square feet, square centimeters, square meters, and so on.

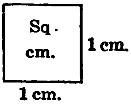


FIG. 427

439. Practical method of estimating area. A practical way to estimate the area of a polygon is to transfer it to squared paper and then count the number of square units inclosed within the figure. Thus, in Fig. 428, if one of the centimeter squares is taken as the unit of measure, the area of the rectangle is 8 sq. cm.

If the bounding lines cut the squares, it becomes necessary to estimate the fractional parts of squares. In general, it is sufficiently accurate to count any square which is evenly divided by the bounding line as a half, to reject any square which lies less than one half within the boundary, and to count as a whole square any square which lies more than one half within the boundary.

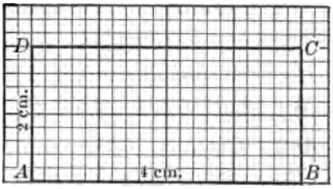


FIG. 428. HOW TO FIND THE AREA OF A RECTANGLE

The approximate areas of many plane figures which are bounded by curved lines may also be found by this method.

440. Equal polygons. Two polygons are said to be equal when their areas are equal. It is evident from our definition of congruent figures (Art. 95) that congruent polygons are equal, but that two equal polygons are not necessarily congruent.

EXERCISES

1. Construct two congruent triangles. Cut them out and place them so as to form a parallelogram.

2. Place the two congruent triangles of Ex. 1 so as to form a kite.

3. Prove that the parallelogram of Ex. 1 and the kite of Ex. 2 are equal but not congruent.

441. Area of a rectangle. Probably the most common illustration of area is that of a rectangle whose sides are integral multiples of some unit of length. Thus, the rectangle $ABCD$ (Fig. 429) is four units long and three units wide, and is divided

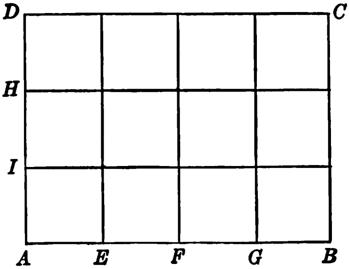


FIG. 429

by lines drawn parallel to the sides into twelve squares, each of which is equal to a unit square.

We therefore say that the area of the rectangle $ABCD$ is twelve square units, or that *its area is equal to the product of the base and altitude.*

We shall assume the truth of the following statements:

442. Theorem. *The area of a rectangle is equal to the product of its base and altitude; that is,* $A = ab$.

443. Corollary 1. *The area of a square is equal to the square of its side.*

444. Corollary 2. *Two rectangles are to each other as the products of their bases and altitudes.*

NOTE. When we say "two rectangles are to each other," we mean "the areas of two rectangles are to each other."

445. Corollary 3. *Two rectangles which have equal bases are to each other as their altitudes.*

446. Corollary 4. *Two rectangles which have equal altitudes are to each other as their bases.*

447. Corollary 5. *The area of a right triangle is equal to one half the product of its legs.*

EXERCISES

1. Prove the following formulas geometrically:

(a) $(x + y)^2 = x^2 + 2xy + y^2$. (b) $(x - y)^2 = x^2 - 2xy + y^2$.

(c) $(x - y)(x + y) = x^2 - y^2$.

2. A square and a rectangle have equal perimeters of 144 in. each. The length of the rectangle is five times its breadth. How do the areas of the square and rectangle compare?

3. On a certain map the linear scale is 2 in. to 50 mi. How many acres are represented by a square $\frac{4}{5}$ in. on a side?

4. The edge of a cube is 5 in. How many square feet are there in its entire exterior surface?

448. Theorem. *The area of a parallelogram is equal to the product of its base and altitude.*

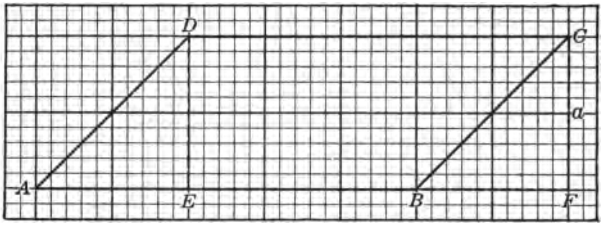


FIG. 430

Given the parallelogram *ABCD*.

To prove that *the area of parallelogram* $ABCD = a \cdot AB$.

Proof. The proof is left to the student.

449. *Corollary 1.* *Two parallelograms are to each other as the products of their bases and altitudes.*

450. *Corollary 2.* *Two parallelograms which have equal altitudes and equal bases are equal.*

451. *Corollary 3.* *Two parallelograms which have equal bases are to each other as their altitudes.*

452. *Corollary 4.* *Two parallelograms which have equal altitudes are to each other as their bases.*

453. Base of a triangle. We have seen that a triangle has three altitudes, one for each of the three sides. Therefore any one of the three sides of a triangle may be taken as its base, and the perpendicular to it from the opposite vertex may be taken as the corresponding altitude. This will be an important point to consider when we come to finding the area of a triangle in the following article and in the exercises which follow. The same point holds with reference to the base and altitude of a parallelogram or trapezoid.

454. Theorem. *The area of a triangle is equal to one half the product of its altitude and its base.*

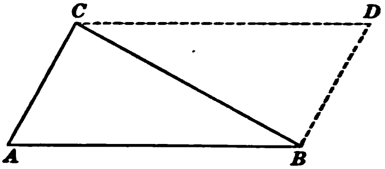


FIG. 431

Given the triangle *ABC*.

To prove that *the area of* $\triangle ABC = \frac{1}{2}ab$.

Proof. The proof is left to the student.

455. Corollary. *If T and T' are the areas of two triangles with altitudes a and a' and bases b and b', then*

1. $\dfrac{T}{T'} = \dfrac{ab}{a'b'}$.

2. *If* $a = a'$, $\dfrac{T}{T'} = \dfrac{b}{b'}$.

3. *If* $b = b'$, $\dfrac{T}{T'} = \dfrac{a}{a'}$.

4. *If* $a = a'$ *and* $b = b'$, $T = T'$.

456. Theorem. *The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.*

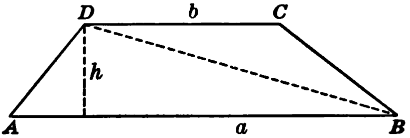


FIG. 432

Given the trapezoid *ABCD*, with the bases *a* and *b* and the altitude *h*.

To prove that *the area of trapezoid* $ABCD = \frac{1}{2}h(a+b)$.

SUGGESTION. Draw DB. Then prove that the area of $\triangle ABD = \frac{1}{2}ah$ and that of $\triangle DBC = \frac{1}{2}bh$. Add and factor.

457. Corollary. *The area of a trapezoid is equal to the product of its altitude and the line segment which joins the midpoints of its nonparallel sides.*

EXERCISES

1. Find the area of the cross section of the canal represented by Fig. 433.

2. Prove that the straight line which joins the midpoints of the bases of a trapezoid divides the trapezoid into two equal trapezoids.

3. Find the lower base of a trapezoid whose altitude is 92 ft., whose base is 84 ft., and whose area is 6164 sq. ft.

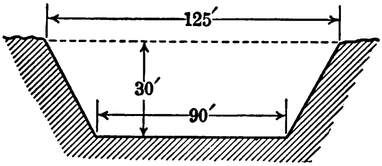


FIG. 433

4. Draw a straight line through a given point on one side of a parallelogram such that the parallelogram is divided into two equal parts.

5. Construct a parallelogram which shall be equal to a given trapezoid and shall have the same altitude as the trapezoid.

6. Draw a straight line perpendicular to the bases of a trapezoid such that it divides the trapezoid into two equal parts.

458. Area of polygons in general. It is difficult, if not impossible, to express the area of an irregular polygon of four or more sides in the same way as that in which we have expressed the area of the triangle, parallelogram, and trapezoid; that is, in terms of its base and altitude. In the general polygon the area is a function of the sides and the angles formed by the sides, and is found by trigonometric methods. In certain special cases, however, the area of a polygon may be found by the preceding geometric formulas. The method consists in dividing the polygon into triangles, parallelograms, and trapezoids by means of diagonals or other lines, and then finding the total area by adding the areas of the several parts. From a practical point of view this method is important.

EXERCISES

1. Draw Fig. 434 to any convenient larger scale and by proper measurements compute the area of the figure.

2. Fig. 435 shows how the area of an irregular polygon may be found, provided the distance of each vertex from a given base line *MN* is known. The distances *AA'*, *BB'*, *CC'*, · · ·, which are called *offsets*, are the bases of trapezoids whose known altitudes are *A'B'*, *B'C'*, · · · respectively. The area *ABCDEF* may be found by performing the necessary additions and subtractions. Assuming that each square represents 1 rd., find the area of the figure.

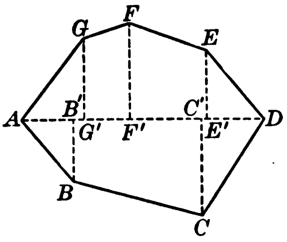


FIG. 434

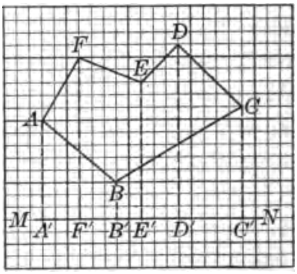


FIG. 435

3. Plot the points whose coördinates are given in the following table, connect them in order by straight lines, and find the area of the resulting polygon :

A	*B*	*C*	*D*	*E*	*F*	*G*
2, 0	3, 4	7, 6	5, 4	6, 6	1, 4	9, 2

459. Trapezoidal rule. The approximate area bounded by a curved line and a straight line, or that of a section cut from such a figure by a pair of perpendiculars from the curved line to the straight line, may be found by a method which is much used by engineers. The following example will make the method clear:

In order to find the flow of water in the stream which is shown in Fig. 436, the area of the cross section must be known. By measuring the depth of the water at equal distances apart (x in the figure) on a line at right angles to the stream, the bases of figures which are approximate trapezoids are found. Then, from the area of the cross section (the sum of the areas of the trapezoids) and the rate of the current, the amount of water which flows past a given point in a given time is determined.

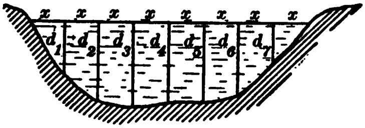


FIG. 436

This method of approximating the area of a figure bounded on one side by an irregular line is called the *trapezoidal rule.* It may be stated more concisely as follows:

Multiply the common distance between the offsets (x) by one half the sum of the first and last offsets added to the sum of the other offsets.

EXERCISES

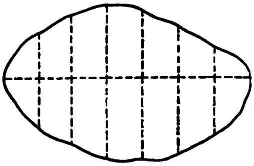


FIG. 437

1. By means of Fig. 437 show that the trapezoidal rule may be employed to find the approximate area of a plane figure bounded by a closed curved line.

2. The nonparallel sides AD and CB of a trapezoid $ABCD$ are each 36 in., and the sides AB and DC are 40 in. and 18 in. respectively. Find the area of the trapezoid $ABCD$

3. Give a complete description of the method of finding the area of the field shown in Fig. 438, using the letters given in the figure.

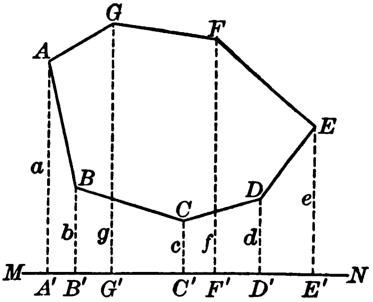


FIG. 438

4. At a price of $120 per acre a man sold a strip of farm land which adjoined a river, as shown in Fig. 439. The strip was surveyed as follows: A base line *AH* was run and stakes were set at intervals of 150 ft. at *B*, *C*, · · ·, *G*. From each of these stakes lines were run to the river at right angles to *AH*. The distances from the river to the points on the base line were as follows: to *B*, 220 ft.; to *C*, 500 ft.; to *D*, 650 ft.; to *E*, 475 ft.; to *F*, 275 ft.; to *G*, 200 ft. Find the area of the strip and the amount received from the sale.

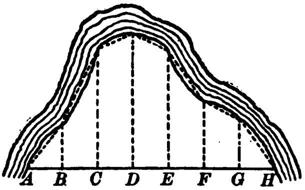


FIG. 439

5. From some map draw the outline of a lake and determine its area by means of the trapezoidal rule and the scale to which the map is drawn.

6. Find the area of the state of Indiana by using a map and the trapezoidal rule. Check your result from a geography.

7. Transfer the leaves shown in Fig. 366 to squared paper and find their approximate areas.

460. General angle. If we think of an angle as the amount of rotation made by a line which rotates in a plane about a fixed point, then we must speak of angles as having direction as well as magnitude. All counterclockwise rotations are considered positive and all clockwise rotations are considered negative. The rotating line in its initial position is called the *initial line*, or *initial side*, and the line in its final position is called the *terminal line.*

EXERCISES

1. By means of the protractor combine the following angles geometrically: $+20°, +45°$; $+80°, -50°$; $-30°, -60°$.

2. What angle will be generated by a spoke of a wagon wheel which is 4 ft. in diameter when the wagon has gone a linear distance of $1257\frac{1}{7}$ ft.? (Use $3\frac{1}{7}$ for π.)

461. Trigonometric functions of an obtuse angle. After the definitions in Art. 460 are thoroughly understood, we can extend our notion of sines, cosines, and tangents to include obtuse angles as well as acute angles. In Fig. 440 let the $\angle XOP$ be the given angle, and let it be either acute, as shown in I, or obtuse, as shown in II. Then, if we call the line XX' the x-axis (the axis of abscissas) and the line YY' the y-axis (the axis of ordinates), distances to the right of the y-axis are $+$, those to the left are $-$, distances above the x-axis are $+$, and those below are $-$.

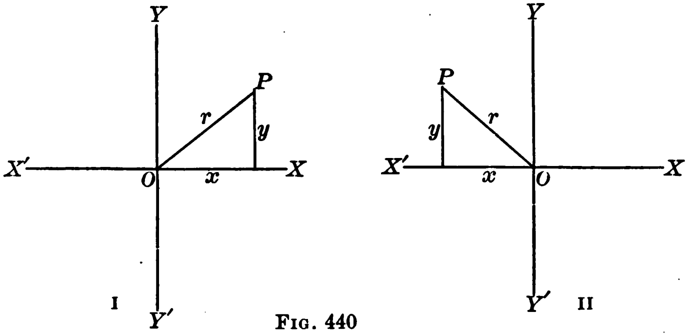


FIG. 440

The distance from the origin to the point P (Fig. 440) on the terminal line is called the *radius vector* of the point P. The radius vector, denoted by r, is always considered positive and such that $r = +\sqrt{x^2 + y^2}$.

We may now define the sine, cosine, and tangent of the $\angle XOP$ as follows:

1. $\sin XOP = \dfrac{\textit{ordinate of the point}}{\textit{radius vector}} = \dfrac{y}{r}$.
2. $\cos XOP = \dfrac{\textit{abscissa of the point}}{\textit{radius vector}} = \dfrac{x}{r}$.
3. $\tan XOP = \dfrac{\textit{ordinate of the point}}{\textit{abscissa of the point}} = \dfrac{y}{x}$.

It should now be clear from the preceding discussion that the sine, cosine, and tangent of any acute angle are each positive.

However, if the angle is obtuse, as in Fig. 440, II, we see that

$$\sin XOP = \frac{+y}{+r} = +\frac{y}{r} \text{(a positive number)};$$

$$\cos XOP = \frac{-x}{+r} = -\frac{x}{r} \text{(a negative number)};$$

and

$$\tan XOP = \frac{+y}{-x} - = \frac{y}{x} \text{(a negative number)}.$$

From the preceding discussion the following statements should now be clear:

1. The sine of the obtuse $\angle XOP$ is identical with the sine of its supplementary acute $\angle X'OP$.
2. The cosine of the obtuse $\angle XOP$ is numerically equal to the cosine of its supplementary acute $\angle X'OP$, but is algebraically opposite in sign.
3. The tangent of the obtuse $\angle XOP$ is numerically equal to the tangent of its supplementary acute $\angle X'OP$, but is algebraically opposite in sign.

EXERCISE

Using the definitions above, find the sine, cosine, and tangent of 150°; of 120°; of 135°.

462. Theorem. *The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides.*

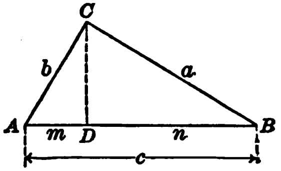


FIG. 441

Given the right triangle *ABC*, with its right angle at *C*.

To prove that $c^2 = a^2 + b^2$.

Proof

STATEMENTS	REASONS
1. Draw $CD \perp AB$ and letter the figure as shown.	1. From a point outside a given line one and only one perpendicular can be drawn to the line.
2. Then $\frac{m}{b} = \frac{b}{c}$, and $\frac{n}{a} = \frac{a}{c}$.	2. If in a right triangle a line is drawn from the vertex of the right angle perpendicular to the hypotenuse, either side about the right angle is a mean proportional between the whole hypotenuse and the segment of the hypotenuse adjacent to it.
3. From 2 we have $b^2 = mc$, and $a^2 = nc$.	3. In a proportion the product of the means equals the product of the extremes.
4. Then $a^2 + b^2 = mc + nc$.	4. By addition.
5. $a^2 + b^2 = c(m + n)$.	5. By factoring.
6. But $m + n = c$.	6. Obvious.
7. $\therefore a^2 + b^2 = c^2$.	7. By substitution.

463. ***Corollary 1.*** *The square of the length of either leg of a right triangle is equal to the square of the hypotenuse minus the square of the other leg.*

464. Various methods of proving the Pythagorean theorem. There are approximately fifty different methods known to the author for proving the Pythagorean theorem stated in Art. 462, and while the student need not know all of these, he may find it interesting to examine a few cases. The following exercises show some of the most interesting of the proofs, Ex. 1, for example, showing how in early times the proof might have been suggested by a study of a tiled floor.

EXERCISES

1. Show how the Pythagorean theorem (Art. 462) may be proved by means of Fig. 442.

2. Prove the Pythagorean theorem by using Fig. 443.

SUGGESTION. Prove that $CBHK = PQGB$ by proving that $CBHK$ is twice as large as $\triangle ABH$ and that $PQGB$ is twice as large as $\triangle GBC$, and then proving that $\triangle ABH = \triangle GBC$. In the same way prove that $AFQP = ACDE$, using $\triangle\triangle ABE$ and ACF.

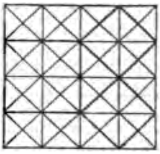


FIG. 442

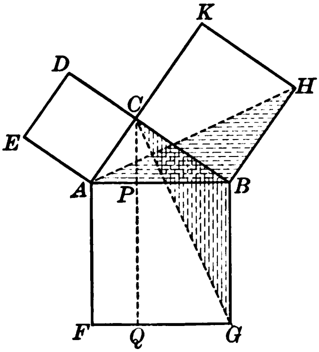


FIG. 443

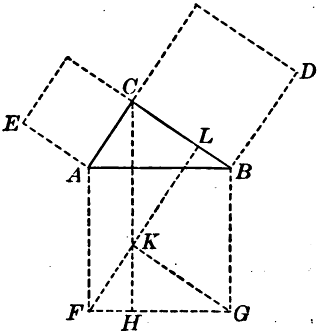


FIG. 444

3. In Fig. 444 FL is drawn parallel to AC, and GK is drawn parallel to AE. Draw CKH and prove that it is parallel to AF. Compare $BCKG$ with the rectangle BH and thus prove the theorem of Pythagoras.

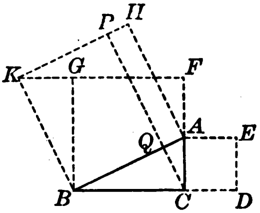


FIG. 445

4. Prove the theorem of Pythagoras by using Fig. 445, where the square $BCFG$ is overlapping the given $\triangle ABC$; that is, it is turned over on the axis BC from left to right.

SUGGESTION. Prove that $BAHK = CDEA + BCFG$. To do this draw KG and prove that KGF is a straight line. Prove that $BCFG = BQPK$ by showing them each twice as large as the $\triangle BCK$. Then prove that $CDEA = QAHP$, and so on.

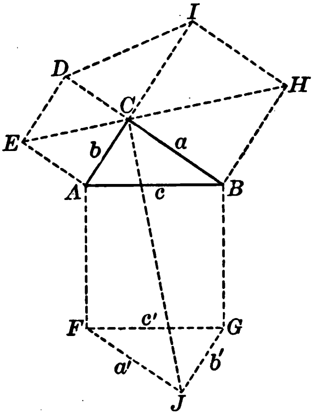


FIG. 446

5. In Fig. 446 the $\triangle FJG$ is constructed congruent to the $\triangle ABC$, and the lines CE and CH are drawn. Prove the Pythagorean theorem from this data.

SUGGESTION. Prove that ECH is a straight line and then show that the quadrilaterals $DEHI$, $EABH$, $CAFJ$, and $CJGB$ are equal.

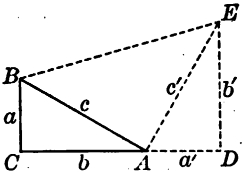


FIG. 447

6. A figure similar to Fig. 447 is used in a proof which has been credited to President Garfield and is known as *Garfield's proof*. On the hypotenuse c one half of a square is constructed, and b' is drawn perpendicular to a'. Then it is proved that △s ABC and ADE are congruent and that the complete figure is a trapezoid of altitude $a' + b$. Complete the proof.

7. In proving the Pythagorean theorem by means of Fig. 448 the student should express a in terms of b and c from his knowledge of the numerical relation of a secant and a tangent. Then he should proceed to show algebraically that $c^2 = a^2 + b^2$. Write out the complete proof.

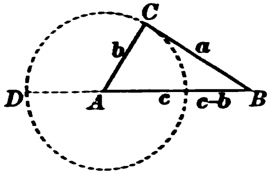


Fig. 448

8. In Fig. 449 we have another case of one square overlapping the given $\triangle ABC$. Construct the square $GJEF$ by drawing GF and FE parallel to AC and BC respectively. Then draw BH and AK perpendicular to AB and show that the right triangles formed are congruent. Complete the proof.

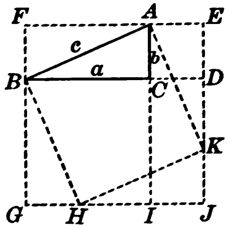


Fig. 449

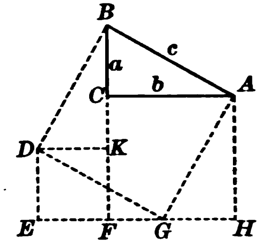


Fig. 450

9. In Fig. 450 $ABDG$ is a square on AB. Through G a line EH is drawn parallel to CA, and the perpendiculars DK, DE, and AH are drawn as shown. Prove that $BC = KF$, that $DKFE$ and $CFHA$ are squares, and that the right triangles formed are equal.

465. Dependence illustrated geometrically. In the preceding exercises the student proved in several ways that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.

In Fig. 451, II, we have the illustration of the preceding fact, C being the right angle of the triangle. Now imagine the $\angle ACB$ to decrease, as shown in Fig. 451, I, and the legs AC and BC to remain constant. Then it is clear that the squares on AC and BC remain constant; but since A and B come nearer together, the square on AB decreases. Therefore, in an acute $\triangle ABC$ the square on the side opposite $\angle C$ is less than the sum of the squares on the other two legs.

In like manner we may increase the size of the $\angle ACB$, as shown in Fig. 451, III, the legs AC and BC and the squares on these legs remaining constant as before. Then, since A and B become farther apart, the square on AB increases. Therefore, in an obtuse $\triangle ABC$ the square on the side opposite the obtuse angle (say $\angle C$) is greater than the sum of the other two sides.

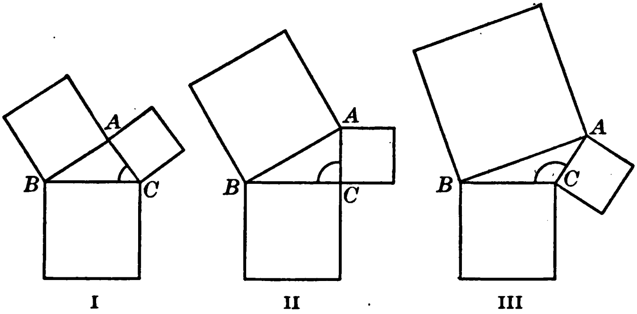


FIG. 451

Thus the size of the square on the side opposite $\angle C$ is seen to depend upon the size of $\angle C$. This is another clear example of the dependence of one quantity upon another.

The student will find that in some texts two more theorems are given to cover the two cases just given, but our knowledge of trigonometry will enable us to find a formula which covers both cases and shows by how much the square on one side of a triangle differs from the sum of the squares on the other two sides.

466. *Theorem.* *In any triangle the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these sides and the cosine of the included angle.*

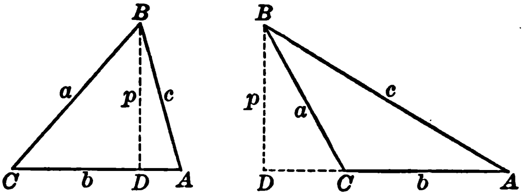


FIG. 452

Given the triangle *ABC*.

To prove that $c^2 = a^2 + b^2 - 2ab \cos C$.

Proof

STATEMENTS	REASONS
1. Draw the altitude from B to CA and call it p.	1. From a point outside a given line one and only one perpendicular can be drawn to the line.
2. Then, in the rt. $\triangle ABD$, $c^2 = p^2 + \overline{AD}^2$.	2. In a right triangle the square on the hypotenuse equals the sum of the squares on the two legs.
3. But $\frac{p}{a} = \sin C$.	3. By definition of sine.
4. From 3, $p = a \sin C$.	4. Multiplying both members of the equation by a.
5. Also, $AD = b - CD$ (CD is negative when $\angle C$ is obtuse).	5. Obvious from the figure.
6. But $\frac{CD}{a} = \cos C$.	6. By definition of cosine.
7. $\therefore CD = a \cos C$.	7. Multiplying both members of the equation in (6) by a.
8. $\therefore AD = b - a \cos C$.	8. By substitution.

Proof (*Continued*)

STATEMENTS	REASONS
9. $\therefore c^2 = (a \sin C)^2 + (b - a \cos C)^2$.	9. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the two legs.
10. From 9, $c^2 = a^2 \sin^2 C + b^2 - 2\,ab \cos C + a^2 \cos^2 C$.	10. By performing the indicated multiplications.
11. $\therefore c^2 = a^2(\sin^2 C + \cos^2 C) + b^2 - 2\,ab \cos C$.	11. By grouping.
12. But $\sin^2 C + \cos^2 C = 1$.	12. Proved in Art. 434.
13. $\therefore c^2 = a^2 + b^2 - 2\,ab \cos C$.	13. By substitution.
14. In like manner, by drawing the other two altitudes to the other sides it can be shown that $a^2 = b^2 + c^2 - 2\,bc \cos A$, and $b^2 = c^2 + a^2 - 2\,ac \cos B$.	

467. The law of cosines. The important relation given in the preceding theorem is known as the *law of cosines.* It is used to solve a triangle (1) when two sides and the included angle are given, or (2) when the three sides are given. The first case can be solved by the theorem of Art. 466. The second is covered by the three parts of the corollary which follows.

468. ***Corollary.*** *If the vertices of a triangle are A, B, C, then*

$$1.\ \cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

$$2.\ \cos B = \frac{a^2 + c^2 - b^2}{2\,ac}.$$

$$3.\ \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

SUGGESTION. In proving this corollary use the three forms of the theorem suggested in Art. 466.

EXERCISES

1. Show that the three formulas in the preceding corollary can be used to find the three angles of a triangle when the three sides are given.

2. If the three formulas in Art. 468 are used to find the three angles of a triangle, show that the check on the accuracy of the work consists in finding out whether the sum of the three angles is 180°.

3. Find the remaining parts of the $\triangle ABC$ if $b = 10$, $c = 8$, and $A = 60°$.

4. Find the three angles of the $\triangle ABC$ if $a = 6$, $b = 8$, and $c = 10$.

5. Find the number of degrees in the greatest angle in the $\triangle ABC$ if $a = 10$, $b = 12$, and $c = 14$.

6. Find the remaining parts of the $\triangle ABC$ if $b = 8$, $c = 5$, and $A = 60°$.

7. Find the three angles of the $\triangle ABC$ if $a = 3$, $b = 7$, and $c = 5$.

469. ***Theorem.*** *If the vertices of a triangle are A, B, C, then*

1. $\angle A$ *is an acute angle if* $a^2 < b^2 + c^2$.
2. $\angle A$ *is an obtuse angle if* $a^2 > b^2 + c^2$.
3. $\angle A$ *is a right angle if* $a^2 = b^2 + c^2$.

SUGGESTION. Use the indirect method of proof.

EXERCISES

1. Show how the preceding theorem illustrates functional relationship; that is, the dependence of one quantity upon another.

2. The sides of a triangle are 8 in., 9 in., and 14 in. What kind of angle is the largest angle of the triangle?

3. Prove the principle known to the Egyptians, that a triangle whose sides are proportional to 3, 4, and 5 is a right triangle.

470. Construction problem. *To construct a triangle when two sides and an angle opposite one of them are given.*

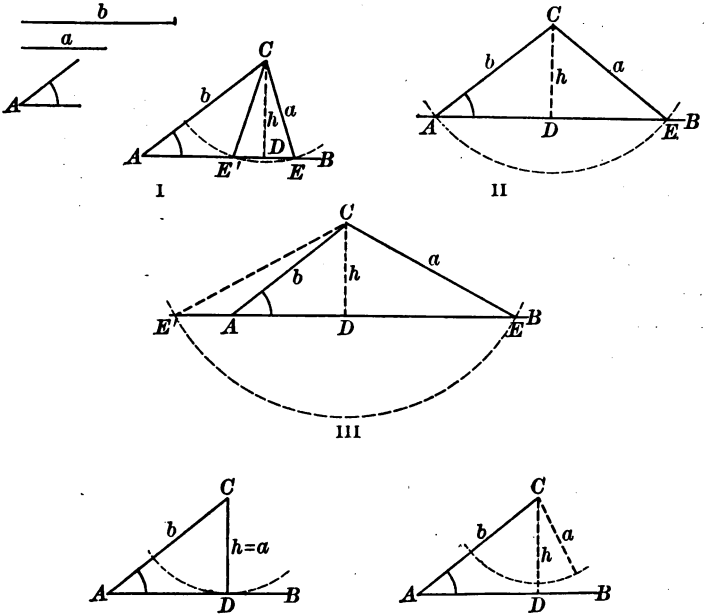


Fig. 453

Given the two sides *a* and *b* and the angle *A* opposite *a*.

Required *to construct the* $\triangle ABC$.

Construction. On a working line AB (Fig. 453) construct an angle equal to the $\angle A$. On the side AC of this angle lay off $AC = b$. With C as a center and a radius equal to a draw an arc of a circle as shown. This arc cuts AB in two points, as in I, II, or III; it just touches AB, as in IV; or it does not cut AB at all, as in V.

Proof. The proof is left to the student.

Discussion. We will consider only the case where $\angle A$ is acute. We then have the following possibilities:

1. Let h be the perpendicular distance from C to AB. Then, if $h < a < b$, the arc cuts AB (Fig. 453) in two points E and E', as

shown in I. Hence there are two possible triangles which fulfill the required conditions; namely, $\triangle ACE$ and $\triangle ACE'$.

NOTE. The expression $h < a < b$ is read "a is greater than h and less than b."

2. If $h < a = b$, the arc cuts AB (Fig. 453) at A and at E, as shown in II. Hence there is only one solution; namely, the $\triangle AEC$, which is isosceles.

3. If $h < a > b$, the arc cuts AB (Fig. 453) in two points E and E', as shown in III. But in this case only one of the triangles thus formed is a solution; namely, the $\triangle ACE$, since the $\triangle ACE'$ does not contain the $\angle A$.

4. If $a = h$, the arc touches AB (Fig. 453), as shown in IV. Then there is only one solution; namely, the $\triangle ADC$, which is a right triangle.

5. If $a < h$, the arc does not cut AB (Fig. 453) at all, as shown in V, and there is no solution.

EXERCISES

Discuss the following cases for the preceding problem:

1. There is no triangle if $a < b \sin A$.

2. There is one right triangle if $a = b \sin A$.

3. There is one isosceles triangle if $b \sin A < a = b$.

4. There are two triangles if $b > a > b \sin A$.

5. There is one triangle if $b < a > b \sin A$.

471. Distance between two given points. If we want to find the distance from the point A to the point B in Fig. 454, we need to know only the coördinates (Art. 211) of the given points. It is evident that by using the Pythagorean theorem we can obtain the following expression for the distance AB:

$$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2}.$$

Then, from the given coördinates we see that

$$AC = 4 - 1 = 3, \quad \text{and} \quad BC = 3 - 1 = 2.$$

Substituting these values of AC and BC, we have

$$AB = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}.$$

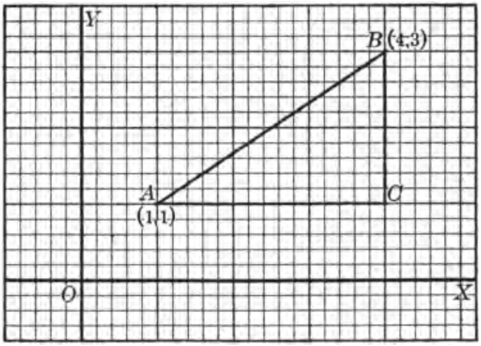


FIG. 454

EXERCISES

1. Determine the distance between each of the following pairs of points:

(a) (1, 3), (2, 6).	(d) (– 5, 3), (– 7, – 6).
(b) (2, 1), (3, 2).	(e) (– 7, – 2), (– 8, – 3).
(c) (– 3, 2), (1, 4).	(f) (4, – 2), (5, 6).

2. Draw the triangle whose vertices are the points (2, 3), (– 1, 4), and (– 5, – 3), and find the lengths of the sides.

3. Show that the points (1, 4), (4, 1), and (5, 5) are the vertices of an isosceles triangle.

4. Show that the distance d from the origin (0, 0) to any point (x, y) is given by the formula $d = \sqrt{x^2 + y^2}$.

5. Show that the points (– 3, – 3), (3, 3), and $(-3\sqrt{3}, 3\sqrt{3})$ are the vertices of an equilateral triangle.

6. Show that the points (4, 4), (2, – 3), (– 7, 1), and (– 9, – 6) are the vertices of a parallelogram.

7. By the method of finding the distance between two points show that the diagonals of any rectangle are equal.

8. Show that the points (0, 0), (– 8, + 15), (– 15, – 8), and (– 23, 7) are the vertices of a square, and find the length of its diagonals.

9. Find the fourth vertex of a parallelogram in which three vertices are the points (– 3, – 3), (4, 1), and (– 7, 5).

472. Distance formula. If the student has solved the preceding exercises with care, he will have no difficulty in proving the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where d is the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, as shown in Fig. 455. This formula, often called the *distance formula*, is used by substituting the known coördinates of two given points, P_1 and P_2, for x_1, x_2, y_1, and y_2. Thus, the distance between the points (4, 3) and (5, – 2) is given by the equation

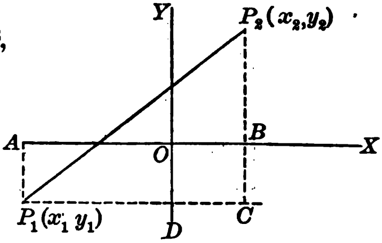


FIG. 455

$$d = \sqrt{(4-5)^2 + (3+2)^2} = \sqrt{1+25} = \sqrt{26}.$$

EXERCISE

Draw a figure to show the distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, and prove the distance formula.

473. Midpoint of a given line segment. In Fig. 456 the student can easily prove that the $\triangle P_1AP_0$ and P_0BP_2 are congruent. We know by Art. 177 that

$$P_1A = \tfrac{1}{2} P_1C,$$

and

$$P_0A = \tfrac{1}{2} P_2C.$$

Then the x value (the abscissa) of P_0 is equal to the abscissa of P_1 + the line P_1A, and the y value (the ordinate) of P_0 is equal to the ordinate of P_1 + the line P_0A. Thus, if $P_0(x_0, y_0)$ represents the midpoint of the line P_1P_2, which joins the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, for all positions of P_1 and P_2 it can be shown that the coördinates of P_0 are given by the formulas

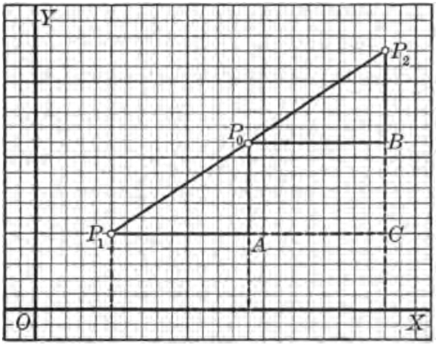


FIG. 456

$$x_0 = \tfrac{1}{2}(x_1 + x_2)$$

and

$$y_0 = \tfrac{1}{2}(y_1 + y_2).$$

These formulas are called the *midpoint formulas.* A more general form of these formulas is given in more advanced mathematical work.

In Fig. 456 it is evident that P_1 is the point (5, 5) and that P_2 is the point (23, 17). Applying the above formulas, we have

$$x_0 = \tfrac{1}{2}(5 + 23) = \tfrac{28}{2} = 14,$$

and

$$y_0 = \tfrac{1}{2}(5 + 17) = \tfrac{22}{2} = 11.$$

Hence P_0 is the point (14, 11), as can be readily verified by counting the squares.

EXERCISES

1. In the midpoint formulas show that the value of x_0 is an arithmetic average of x_1 and x_2 and that y_0 is an arithmetic average of y_1 and y_2.

2. Take the point P_1 in the third quadrant and the point P_2 in the first quadrant, as in Fig. 457. Prove that the midpoint formulas given on page 377 are true for this case.

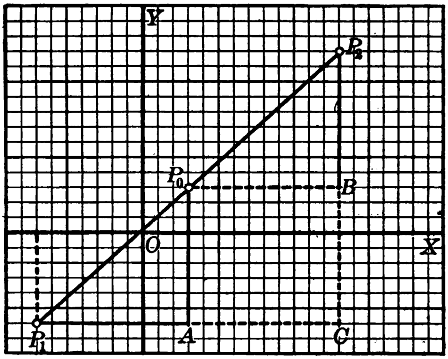


FIG. 457

3. Find the midpoints of the sides of the triangle described in Ex. 2, Art. 471. Draw the medians of the triangle and find their lengths.

4. In the $\triangle ABC$ shown in Fig. 454 find the midpoints of the sides AB and BC and the length of the line joining them. How does it compare with the length of the third side AC?

5. Find the midpoints of the sides of the quadrilateral whose vertices are the points (2, 4), (9, – 5), (0, 20), and (13, 18). Join the midpoints in order by straight lines and prove that the figure thus formed is a parallelogram by showing that the opposite sides are of equal length.

6. By proving that their midpoints coincide, show that the diagonals of the parallelogram of Ex. 5 bisect each other.

474. Locus of the equation $\frac{y}{x} = k$. If we let y and x represent the coördinates of a variable point which moves so that the ratio of the ordinate y to the abscissa x is equal to a constant k, then the problem of finding the locus of $\frac{y}{x} = k$ is the problem of finding the locus of all points whose coördinates satisfy the equation $\frac{y}{x} = k$. We can construct the locus as follows:

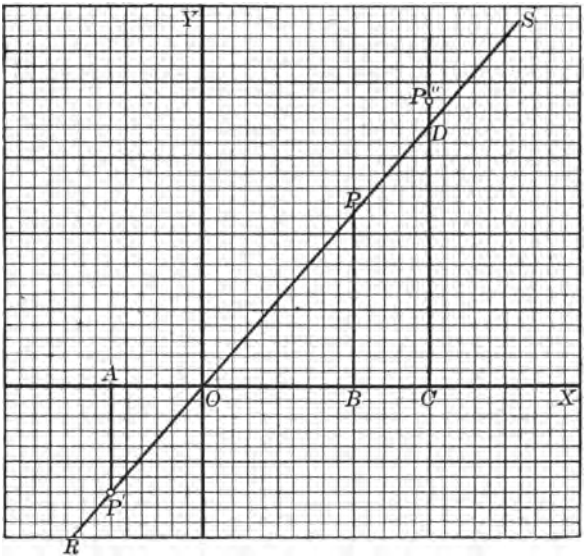


FIG. 458

Construction. Locate one point of the locus by letting $x = 1$ and $y = k$ (that is, y = any number). Draw the line RS (Fig. 458) through this point, which we may denote by P, and the origin (0, 0), which is obviously another point of the locus. Then RS is the required locus.

Analysis. In order to prove that RS is the locus of $\frac{y}{x} = k$, we must prove that any point on RS has coördinates which satisfy the equation $\frac{y}{x} = k$ and that any point not on the line RS has coördinates which do not satisfy the equation $\frac{y}{x} = k$ (see Art. 193).

Proof

STATEMENTS	REASONS
1. Let P' (x', y') be any point on RS other than P. Draw AP' and $PB \perp$ the x-axis.	1. From a point outside a given line one and only one perpendicular can be drawn to the line.
2. In the $\triangle OPB$ and $AP'O$, $\angle AOP' = \angle POB$.	2. Vertical angles are equal.
3. $\angle P'AO = \angle PBO$.	3. All right angles are equal.
4. $\therefore \triangle OPB \backsim \triangle AP'O$.	4. Two triangles are similar if two angles of one are equal to the two homologous angles of the other.
5. $\therefore \frac{y'}{x'} = \frac{k}{1}$.	5. Homologous sides of similar triangles are in proportion.
6. $\therefore$ any point on RS has coordinates which satisfy the equation $\frac{y}{x} = k$.	6. Obvious, since the point P was taken as any point of the line RS.
7. Let P'' (x'', y'') be any point not on RS, and let the perpendicular $P''C$ (y'') cut RS at D.	7. By agreement.
8. Then, since D is on RS, $\frac{DC}{x''} = k$.	8. In the same manner as proved in 4 and 5.
9. $\therefore \frac{y''}{x''} \neq \frac{DC}{x''}$, unless $y'' = DC$, which clearly is not the case.	9. Obvious.
10. $\therefore$ the coördinates of P'' do not satisfy the equation $\frac{y}{x} = k$.	
11. $\therefore$ the line RS is the locus of $\frac{y}{x} = k$.	

475. *Corollary 1.* *The locus of the equation $y = kx$ is a straight line through the origin and the point $(1, k)$.*

476. *Corollary 2.* *The locus of the equation $y = kx + b$ is a straight line.*

Suggestion. The equation $y = kx + b$ states that the value of y is always b units more than the value of y in the equation $y = kx$. Since the locus of the equation $y = kx$ is a straight line, the locus of the equation $y = kx + b$ is a straight line, which is parallel to the locus of the equation $y = kx$, and either above or below it according as b has a positive or negative value.

EXERCISES

Draw the graph for each of the following:

1. $y = \frac{2}{3}x$.

2. $y = 5x$.

3. $y = 2x$.

4. $y = 2x + 4$.

5. $y = \frac{1}{2}x + 1$.

6. $y = -2x + 3$.

7. The cost of sending a package by parcel post into a certain zone is given by the equation $y = 2x + 3$ (two cents per ounce plus three cents). Graph the equation and find y when x is 12 oz.

477. Inclination. The *inclination* of a line is the angle made by the line with the x-axis, the x-axis being regarded as the initial side of the angle. If the line is parallel to the x-axis, its inclination, of course, is 0. We denote the inclination of a line by the Greek letter α.

478. Slope. The tangent of the inclination of a line is called the *slope* of the line and is denoted by k; that is, $\tan \alpha = k$. If k is positive, then α is acute; if k is negative, then α is obtuse; and conversely. When the equation of the line is in the form $y = kx$ or $y = kx + b$ the slope of the line is equal to k. To find the slope it may be necessary to write the equation in the form $y = kx + b$. For example, if $3x - y = 5$, then $y = 3x - 5$.

479. Theorem. *In any triangle the three sides are proportional to the sines of the opposite angles.*

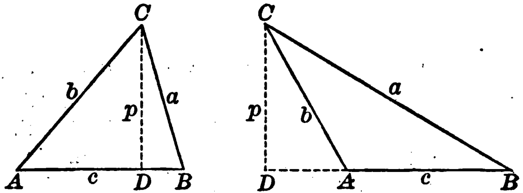


FIG. 459

Given the triangle *ABC*.

To prove that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Proof

STATEMENTS	REASONS
1. In Fig. 459, I, the $\angle A$ is acute; in Fig. 459, II, the $\angle A$ is obtuse.	1. Obvious.
2. Draw $CD \perp AB$ and denote CD by p.	2. From a point outside a given line one and only one perpendicular can be drawn to the line.
3. Then $\sin A = \dfrac{p}{b}$, and $\sin B = \dfrac{p}{a}$.	3. By definition of sine.
4. $\therefore \dfrac{\sin A}{\sin B} = \dfrac{a}{b}$.	4. By dividing the first equation of 3 by the second.
5. $\therefore \dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.	5. By first alternating the means in 4 and then inverting the result.
6. In like manner, by drawing an altitude from A to BC, we have $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.	6. As proved in 1–5.
7. $\therefore \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.	7. Things equal to the same thing are equal to each other.

480. The law of sines. The important relation expressed by the preceding theorem is known as the *law of sines.* It is used to solve a triangle (1) when two angles and a side are given and (2) when two sides and an angle opposite one of these sides are given. In the second case two solutions of the problem are possible, as the student will see from a scale drawing which shows the given facts, or from a study of the solutions of the following exercises.

EXERCISES

Using the law of sines, solve the following exercises:

1. If $c = 150$, $A = 50°$, and $B = 62°$ in $\triangle ABC$, find b, a, and C.

2. If $A - 60°$, $a = 4.56$, and $b = 3.24$ in $\triangle ABC$, find B, C, and c.

481. Triangulation. Surveyors often use a process called *triangulation* when they are mapping large areas. By this process they are able to locate accurately a group of widely scattered points which they can use as the basis for the ordinary surveys of small areas.

A base line AB (Fig. 460) is first laid off and accurately measured. Then with the transit the $\angle s\ BAC$ and ABC to a visible point C are accurately measured. The length of CB is then computed by the law of sines. Then, using the line CB as a base, a point D is selected, the $\angle s\ DBC$ and DCB are measured as before, and the distance CD is computed.

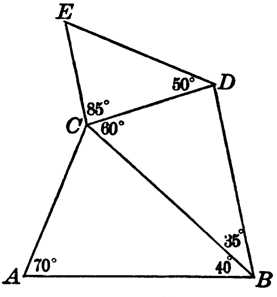


FIG. 460

It is evident that this process can be continued indefinitely by locating the point E, and so on. In this way the surveyors locate a set of points which are connected by a network of triangles, and from any of the computed lines as a base the series of triangles can be extended indefinitely to include all parts of the section which is being mapped. It is obvious that distances, such as *ED*, which may be inaccessible for direct measurement can be accurately measured, and that the areas of large sections of land can be found by this method.

EXERCISES

1. In Fig. 460, if $AB = 2000$ ft. and the angles have the values shown in the figure, find the length of *ED*.

2. With a field protractor make a triangulation covering a small field. Select your final line, such as *ED* in Fig. 460, so that you can check your computation by direct measurement.

482. *Theorem.* *If a, b, and c are the sides of a triangle ABC, s is one half the perimeter, and h_c is the altitude to c, then*

$$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

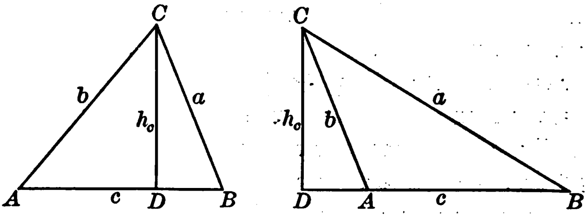


FIG. 461

Given the triangle *ABC*, where *a*, *b*, and *c* are the sides opposite the angles *A*, *B*, and *C* respectively, $s = \frac{1}{2}(a + b + c)$, and h_c is the altitude upon *c*.

To prove that $h_c = \dfrac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.

Proof

Statements	Reasons
1. Draw the altitude h_c (CD) in each triangle.	1. From a point outside a given line one and only one perpendicular can be drawn to the line.
2. In the rt. $\triangle ADC$, $h^2 = b^2 - \overline{AD}^2$.	2. By the Pythagorean theorem.
3. But $AD = b \cos A$.	3. By definition of "cosine."
4. Also $\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}$.	4. By the law of cosines.
5. $\therefore h_c^2 = b^2 - b^2\left(\frac{b^2 + c^2 - a^2}{2\,bc}\right)^2$ $= b^2 - \frac{(b^2 + c^2 - a^2)^2}{4\,c^2}$ $= \frac{4\,b^2c^2 - (b^2 + c^2 - a^2)^2}{4\,c^2}$.	5. By substitution and simplifying as shown.

6. Then $h_c^2 = \frac{\{2\,bc + (b^2 + c^2 - a^2)\}\,\{2\,bc - (b^2 + c^2 - a^2)\}}{4\,c^2}$, by factoring the numerator of the right member.

7. $h_c^2 = \frac{\{(b + c)^2 - a^2\}\,\{a^2 - (b - c)^2\}}{4\,c^2}$

$= \frac{(b + c + a)(b + c - a)(a - b + c)(a + b - c)}{4\,c^2}$, by regrouping and factoring.

8. Now let $a + b + c = 2\,s$.	8. Given.
9. Then $b + c - a = 2\,s - 2\,a$ $= 2(s - a)$.	9. By subtracting $2\,a$ from both sides of 8.
10. In like manner, $a - b + c = 2\,s - 2\,b$ $= 2(s - b)$, and $a + b - c = 2\,s - 2\,c$ $= 2\,(s - c)$.	10. As shown in 9.

11. $\therefore h_c^2 = \frac{2\,s \,.\, 2\,(s - a) \cdot 2\,(s - b) \cdot 2\,(s - c)}{4\,c^2}$, by substitution.

12. From 11, $h_c^2 = \frac{4 \cdot s(s = a)(s - b)(s - c)}{c^2}$, by dividing both numerator and denominator of the right member by 4.

13. $\therefore h_c = \frac{2}{c}\sqrt{s\,(s - a)\,(s - b)\,(s - c)}$, by taking the square root.

483. Corollary 1. *If the notation of the theorem of Art. 482 is used, then in the* $\triangle ABC$*:*

$$1.\ h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$2.\ h_b = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

Suggestion. Prove this corollary by means of symmetry, or otherwise.

484. Corollary 2. *The area of a* $\triangle ABC$ *is given by the formula*

$$T = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, *and* c *are the sides,* $s = \frac{1}{2}(a+b+c)$, *and* T *represents the number of square units in the area.*

Note. The formula in Corollary 2 is known as Hero's formula.

485. Corollary 3. *If* a, b, *and* c *are the sides of a* $\triangle ABC$, *and* T *is the area, then:*

$$1.\ \sin A = \frac{2T}{bc}.$$

$$2.\ \sin B = \frac{2T}{ac}.$$

$$3.\ \sin C = \frac{2T}{ab}.$$

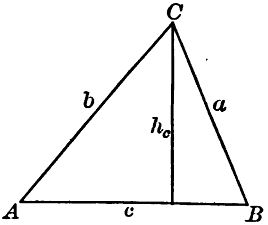


Fig. 462

Suggestion. We know that $\sin A = \frac{h_c}{b}$ by definition of a sine. Then we may substitute the value of h_c obtained in the theorem of Art. 482 and simplify the result. In like manner, $\sin B$ and $\sin C$ may be obtained.

EXERCISES

1. If each of two sides of a triangle is 10 and the included angle is 60°, what is the third side?

Suggestion. Use the law of cosines.

2. If each of two sides of a triangle is 12 and the included angle is 120°, what is the third side?

3. The sides of a triangle are 7, 8, and 10. What kind of an angle is the largest angle of the triangle? Find the number of degrees in this angle.

4. The sides of a triangle are 10, 14, and 20. What kind of an angle is the largest angle of the triangle? Find the number of degrees in this angle.

5. If the sides of a triangle are 7, 8, and 11, find s, $s-a$, $s-b$, and $s-c$.

6. If the sides of a triangle are 7, 8, and 9, find the three altitudes.

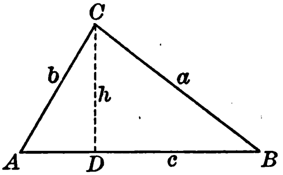


FIG. 463

7. Show that the area of the $\triangle ABC$ (Fig. 463) equals one half the product of any two sides multiplied by the sine of the included angle; that is, show that $T = \frac{bc \sin A}{2}$.

SUGGESTION. Draw the altitude CD, as in Fig. 463.

Then $T = \frac{ch}{2}$. Why?

But by definition, $\frac{h}{b} = \sin A$.

Hence $h = b \sin A$,

and by substitution $T = \frac{bc \sin A}{2}$.

Write out the proof in full.

8. A boy measures the sides of a drug store which completely covers a triangular lot, and finds that the walls extend 60 ft. and 80 ft. respectively from the street corner. He measures the angle between the streets with a field protractor and finds it to be 58°. How can he find the area of the lot? What result should he obtain?

9. Prove that the area of a parallelogram equals the product of two sides and the sine of the angle included between them.

10. Show that the area of an equilateral triangle of side a is equal to $\frac{a^2}{4}\sqrt{3}$.

SUGGESTION. Use Hero's formula (Art. 484).

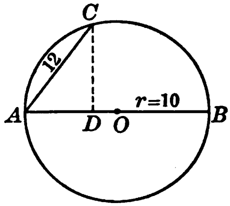


FIG. 464

11. Find the diagonal of a cube each of whose edges is a feet in length.

12. A tree BD which is 70 ft. high is broken by the wind at a point C, and the end D strikes the ground at A, which is 35 ft. from the foot of the tree. Find the length of CD.

13. The radius of the circle shown in Fig. 464 is 10. Find the projection of the chord AC upon the diameter AB.

486. Area of a triangle in relation to the inscribed circle. In finding formulas for the area of a triangle we have thus far used as a fundamental formula the well-known relation $A = \frac{ah}{2}$. It is obvious, however, that this formula lacks symmetry, inasmuch as it uses only one side of the triangle in finding the area.

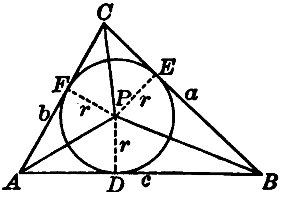


FIG. 465

If we take a point P within a given triangle, as shown in Fig. 465, and join it to the three vertices, then the area of the triangle is the sum of the areas of the three triangles thus formed, and each side is used. More particularly, if the point of intersection of the three bisectors of the angles of the triangle be chosen as the point P, then P is the center of the inscribed circle. Then, letting r represent the radius of the inscribed circle, we find the area of the $\triangle ABC$ as follows:

$$A = \triangle PBC + \triangle PCA + \triangle PAB$$

$$= \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr \qquad \text{Why?}$$

$$= \frac{r}{2}(a + b + c). \qquad \text{Why?}$$

Expressed in words, this formula reads: *The area of a triangle is equal to one half the product of the perimeter and the radius of the inscribed circle.*

487. Finding the radius of the inscribed circle. If in the equation $A = \frac{r}{2}(a+b+c)$ of Art. 486 we let $2s = a + b + c$, the equation for the area of the triangle becomes

$$A = sr.$$

In Art. 484 we have already proved that

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore sr = \sqrt{s(s-a)(s-b)(s-c)},$$

and

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s},$$

or

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

This gives us a formula for the radius of the inscribed circle in terms of the three sides of the given triangle.

EXERCISES

1. In Fig. 465 show that $AF = AD$, $BD = BE$, and $CE = CF$.

2. In Fig. 465 show that $AD + BE + CF = s$ and show that $AD = AF = s - a$. Similarly, show that $BE = BD = s - b$ and that $CF = CE = s - c$.

3. A triangular city lot has sides of 100 ft., 80 ft., and 60 ft. respectively. What is the radius of the largest cylindric gasometer that can be placed on this lot?

488. Area of a triangle in relation to the circumscribed circle. We know from Art. 196 that the center of the circumscribed circle of a triangle is the point of intersection of the three perpendicular bisectors of the sides.

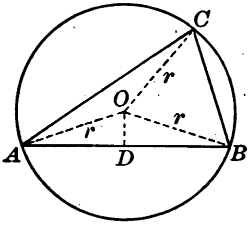


FIG. 466

If D (Fig. 466) is the midpoint of AB, then OD is perpendicular to AB because O is the point where the three perpendicular bisectors of the sides meet. Then $\angle AOB$ is measured by $\overset{\frown}{AB}$ and $\angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$, since the first is a central angle and the second is an inscribed angle. Also, $\angle AOD$ is measured by $\frac{1}{2}\overset{\frown}{AB}$, since it is a central angle and OD produced bisects $\overset{\frown}{AB}$. (Why?) Thus it is clear that $\angle AOD = \angle ACB$, since each angle is equal to $\frac{1}{2}\angle AOB$; that is, $\angle AOD = \angle C$, an angle of the given triangle.

Then in the rt. $\triangle AOD$,

$$\sin C (\text{or } \sin AOD) = \frac{AD}{AO} = \frac{\frac{c}{2}}{r} = \frac{c}{2r},$$

or

$$2r = \frac{c}{\sin C}.$$

But by the law of sines (Art. 479),

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Hence

$$2r = \frac{a}{\sin A},$$

or

$$\sin A = \frac{a}{2r}.$$

By substituting this value of $\sin A$ in the equation

$$A = \tfrac{1}{2}\, bc \sin A,$$

which is deduced from Corollary 3, Art. 485, we have

$$A = \frac{abc}{4r}.$$

This gives us a formula for the area of a triangle in terms of the three sides and the radius of the circumscribed circle.

EXERCISES

1. If the sides of a triangle are 6, 12, and 14, find the area of the triangle and the radius of the circumscribed circle.

2. Express the area of a triangle in terms of the diameter of the circumscribed circle.

3. If the sides of a triangle are 7, 13, and 15, find the area and the diameter of the circumscribed circle.

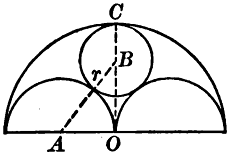


FIG. 467

4. If the sides of a triangle are 4, 4, and 8, find the area, the angles, and the radii of the inscribed and circumscribed circles.

5. Fig. 467 shows half of a circular church window, the radius of which is 12 ft. Find the radius of the smallest circle.

6. If in $\triangle ABC$, $a = 4$, $b = 7$, and $c = 9$, find the length of the radius of the inscribed circle and of the line segments into which this circle divides the sides of the triangle.

7. The radius of a circle inscribed in an equilateral triangle is equal to one third of the altitude.

8. If the inscribed circle bisects one side of a triangle, what relation will exist between its sides? If it has one of the sides tangent at a trisection point?

9. The hypotenuse of a right triangle is equal to the sum of the legs decreased by twice the radius of the inscribed circle.

489. *Construction problem.* *To transform a given quadrilateral into an equivalent triangle.*

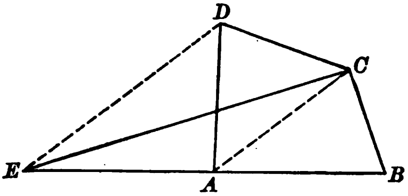


FIG. 468

Given the quadrilateral *ABCD*.

***Required** to transform ABCD into an equivalent triangle.*

Construction. Draw the diagonal AC. Draw $DE \parallel AC$ and meeting BA produced at E, and draw EC.

Then $\triangle ECB$ is the required triangle.

Proof

STATEMENTS	REASONS
1. $\triangle DCA = \triangle EAC$.	1. Triangles having equal bases and equal altitudes are equal.
2. $\triangle DCA + \triangle CAB$ $= \triangle EAC + \triangle CBA$.	2. Equals added to equals give equals.
3. $\therefore \triangle ECB = ABCD$.	3. Obvious.

EXERCISES

1. In how many ways may a quadrilateral be transformed into an equivalent triangle?

2. To transform a square into an equivalent right triangle; into an equivalent isosceles triangle.

3. Any median of a triangle divides the triangle into two equivalent triangles.

4. The diagonals of a parallelogram divide the parallelogram into two pairs of equivalent triangles.

490. *Construction problem.* *To transform a given polygon into an equivalent triangle.*

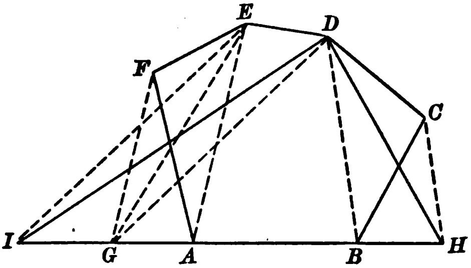


FIG. 469

Given the polygon *ABCDEF*.

Required *to transform $ABCDEF$ into an equivalent triangle.*

Construction. Draw DB and construct $\triangle BDH = \triangle BCD$, as in Art. 489. In like manner, construct $\triangle AGE = \triangle AFE$. Then $GEDH = ABCDEF$. In like manner, construct $\triangle IDH = GHDE$. Then $\triangle IDH = ABCDEF$.

Proof. The proof is left to the student.

EXERCISES

1. Transform a given pentagon into an equivalent triangle. Measure the base and altitude of the triangle and find its area. What is the area of the pentagon?

2. Construct a square equal to a given quadrilateral.

3. Construct a triangle equivalent to a given trapezoid and having the same altitude as the trapezoid.

4. Construct a parallelogram equivalent to a given trapezoid and having the same altitude as the trapezoid.

5. Divide a trapezoid into two equivalent parts by drawing a straight line perpendicular to the bases of the trapezoid.

6. Construct an isosceles triangle equivalent to a given triangle and on the same base.

491. *Construction problem.* *To construct a square equal to the sum of two or more given squares.*

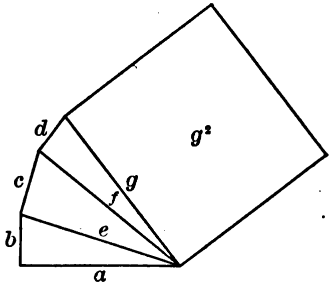


FIG. 470

Given a, b, c, . . ., = the sides of given squares.

Required *to construct a square equal to the sum of the given squares.*

NOTE. Since the construction and proof may be easily inferred from Fig. 470, they are left to the student.

EXERCISES

1. Construct a square equal to three times a given square.

2. Show how Fig. 471 furnishes a method of constructing the square root of any number. Is this a practical method? Give reasons for your answer.

3. What other methods do you know for finding the square roots of numbers? Illustrate each method.

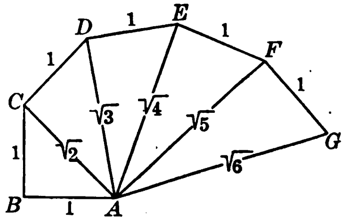


FIG. 471

4. Can a figure be constructed that is equivalent to the sum of a square and a pentagon? Explain your answer

492. Construction problem. *To construct a square equal to the difference of two given squares.*

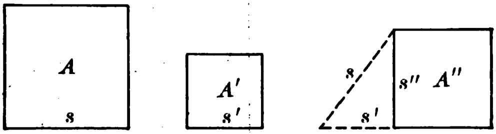


FIG. 472

Given the squares A and A'.

Required *to construct a square equal to $A - A'$.*

NOTE. The construction and proof are left to the student.

EXERCISE

Prove that the area of an isosceles right triangle is equal to one fourth the area of the square upon the hypotenuse.

493. Theorem. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*

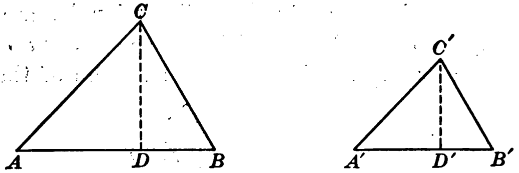


FIG. 473

Given the similar triangles ABC and $A'B'C'$.

To prove that

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}.$$

Proof

STATEMENTS	REASONS
1. Draw the altitudes CD and $C'D'$.	1. From a point outside a given line one and only one perpendicular can be drawn to the line.

Proof (*Continued*)

STATEMENTS	REASONS
2. $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \cdot CD}{A'B' \cdot C'D'}$.	2. Two triangles are to each other as the products of their bases and altitudes.
3. But $\frac{CD}{C'D'} = \frac{AB}{A'B'}$.	3. Homologous sides of similar triangles are to each other as their altitudes.
4. $\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$.	4. By substituting $\frac{AB}{A'B'}$ for $\frac{CD}{C'D'}$ in 2.
5. In like manner, $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$.	5. By a similar reasoning.

EXERCISES

1. Verify the truth of the preceding theorem by counting the little triangles in the large triangles of Fig. 474.

2. Prove that the area of two similar triangles are to each other as any two homologous altitudes; as any two homologous medians; as any two homologous bisectors of the angles.

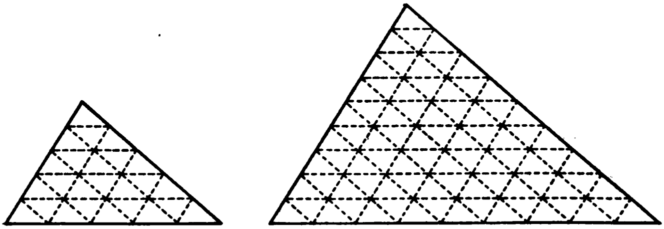


FIG. 474

3. Show that the area of any two similar triangles are to each other as any two homologous lines of the triangles.

4. In the similar $\triangle\triangle ABC$ and $A'B'C'$, if $AC = 2A'C'$, how does the area of $\triangle ABC$ compare with that of $\triangle A'B'C'$? Illustrate by means of a figure.

5. What is the ratio of $\triangle ABC$ to $\triangle A'B'C'$ if $AC = A'C'$? if $AC = 4\,A'C'$? if $AC = \frac{3}{4}\,A'C'$?

6. The length of one side of a triangular garden patch containing $1\frac{1}{2}$ A. is 12 rd. What is the area of a similar triangular patch whose homologous side is 30 rd. long?

494. *Theorem.* *The areas of two similar polygons are to each other as the squares of any two homologous sides.*

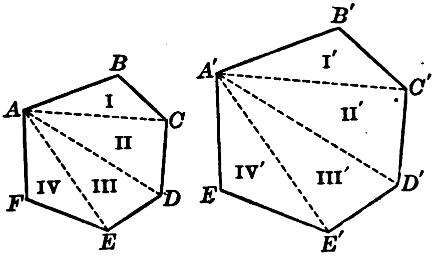


FIG. 475

Proof. The proof is left to the student.

EXERCISES

1. If similar polygons are constructed on the three legs of a right triangle, show that the one constructed on the hypotenuse equals the sum of the other two.

2. Show that the square on the altitude of an equilateral triangle is equal to three fourths of the square on a side.

3. Bisect a given triangle by a straight line drawn through a vertex.

4. Draw a line from a vertex of a triangle to a point in the opposite side such that the triangle is divided into two triangles which have the ratio $\frac{1}{2}$; the ratio $\frac{2}{3}$.

5. Divide a parallelogram into three equal parts by lines drawn from one vertex.

495. *Construction problem.* *To construct a square equal to a given rectangle.*

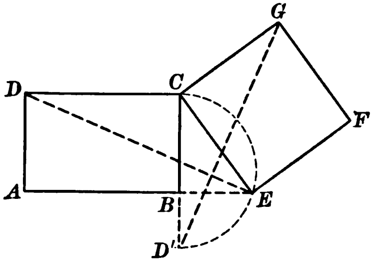


FIG. 476

Given the rectangle *ABCD*.

Required *to construct a square equal to $ABCD$.*

Construction. Produce CB to D' making $CD' = AB$, and on CD' as a diameter construct a semicircle. Construct $BE \perp CD'$, meeting the semicircle at E, and draw CE.

Then the square constructed on CE is the required square.

Proof

STATEMENTS	REASONS
1. Draw DE and GD'.	1. One and only one straight line can be drawn between two points.
2. Then $\triangle DEC \cong \triangle CD'G$.	2. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.
3. But $ABCD = 2\triangle DCE$, and $CEFG = 2\triangle CD'G$.	3. Because a triangle is one half a rectangle which has the same base and altitude.
4. $\therefore ABCD = CEFG$.	4. Things equal to the same thing are equal to each other.

496. Construction problem. *To construct a square equal to a given parallelogram.*

497. Construction problem. *To construct a square equal to a given triangle.*

498. Construction problem. *To construct a square equal to a given polygon.*

EXERCISES

1. Construct a square which contains 20 sq. cm.

2. Construct a square equal to three fifths of a square which is 9 cm. on a side.

3. Construct a square equal to four sevenths of a given square.

4. Construct a square equal to the difference of two given pentagons.

5. Construct a square equal to the sum of two given triangles.

6. Construct a square equal to three fourths of a given pentagon.

7. Construct a right triangle equal to a given square.

8. Construct a square which is twice a given square; twice a given parallelogram; twice a given triangle.

9. Construct a parallelogram equal to a given rectangle and having a given line segment as base.

SUGGESTION. Let x represent the required altitude of the parallelogram, and let a, b, and c represent the altitude of the given rectangle, the base of the given rectangle, and the base of the parallelogram respectively. Then $x = \frac{ab}{c}$.

10. Construct a rectangle equal to a given rectangle and having a given line segment as base.

11. Construct a triangle equal to a given triangle and having a given line segment as base.

12. Divide a given triangle into two equal parts by a line parallel to the base.

13. Trisect a parallelogram by two straight lines drawn through one of its vertices.

14. Trisect a given triangle by straight lines drawn through a vertex.

15. Divide a given triangle into any number of equal parts by straight lines drawn through a vertex.

16. Construct a rectangle having a given base and equal to three fourths of a given square.

17. Construct a rectangle of given base equal to the difference between two given squares.

18. On the same base with a given parallelogram, and having one of its sides equal to a given straight line, construct a parallelogram equal to the given parallelogram.

19. Construct a triangle having a given base and equal to a given parallelogram.

20. Construct a parallelogram having a given altitude and equal to a given triangle; equal to a given square; equal to a given trapezoid.

21. Construct a triangle having a given altitude and equal to a given trapezoid.

22. Construct a square equal to a given trapezoid.

23. Construct a parallelogram equivalent to a given trapezoid and having an altitude equal to a given line segment.

24. Construct a right triangle having a given altitude and equivalent to a given parallelogram.

25. Bisect the area of a triangle by drawing a line through a given point in one of the sides.

CHAPTER XI

INSCRIBED AND CIRCUMSCRIBED REGULAR POLYGONS

499. Regular polygons in nature and in designs. We find many instances of regular polygons in nature and in ornamental designs, such as hexagonal honeycombs, lace patterns, linoleum patterns, tile floors, and so on. The figures above (see Fig. 477) will help the student to understand some of the uses made of regular polygons.

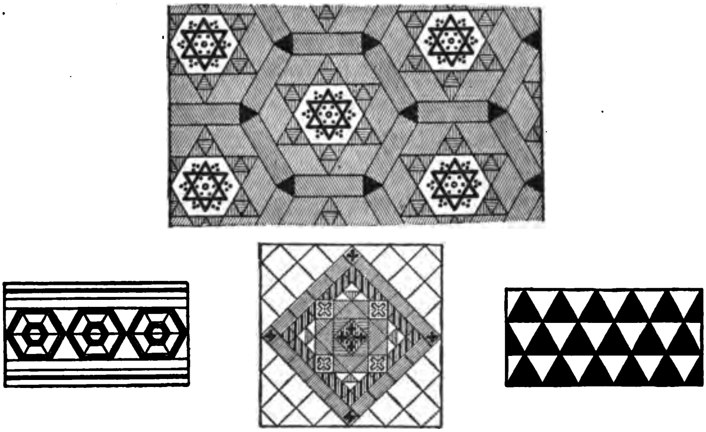


FIG. 477. HOW REGULAR POLYGONS ARE USED

500. Inscribed polygon. An inscribed polygon (see Art. 199) is a polygon all of whose vertices lie on a circle. The circle is said to be circumscribed about the polygon. In Fig. 478 $ABCDE$ is an inscribed pentagon.

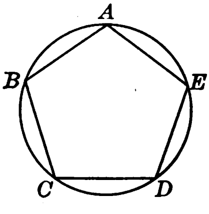


FIG. 478

501. Circumscribed polygon. A *circumscribed polygon* is a polygon whose sides are tangent to a circle. The circle is said to be *inscribed in the polygon.* In Fig. 479 $ABCDE$ is a circumscribed pentagon.

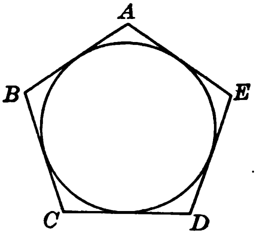


FIG. 479

502. Center of a regular polygon; radius; apothem. The common center O (Fig. 480) of the inscribed and circumscribed circles is called the *center* of the regular polygon. The radius R of the circumscribed circle is called the *radius* of the regular polygon. The radius r of the inscribed circle is called the *apothem* of the regular polygon. Obviously the apothem is the perpendicular from the center to a side of a regular polygon.

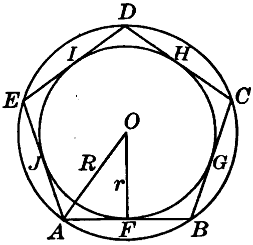


FIG. 480

EXERCISES

1. Show that an equilateral triangle is necessarily equiangular and hence is regular.

2. Draw freehand a quadrilateral that is

(a) Equilateral but not equiangular.
(b) Equiangular but not equilateral.
(c) Neither equilateral nor equiangular.
(d) Both equilateral and equiangular.

3. Which of the quadrilaterals in Ex. 2 is regular? What is a regular quadrilateral called?

4. Show how a polygon of more than three sides may be equilateral without being equiangular; equiangular without being equilateral.

5. Show that the exterior angles at the vertices of a regular polygon are equal.

6. What is the number of degrees in each angle of a regular polygon of three sides? four sides? five sides? six sides? eight sides? ten sides? twelve sides? n sides?

7. In making tile floors what kinds of regular polygons can be used such that only tiles of the same shape are needed to fill the space about a given point?

8. Show that the designs of Fig. 477 are made up of regular polygons and show why they fit.

9. What is the perimeter of a regular hexagon one of whose sides is 7 in.? of a regular octagon one of whose sides is 5 in.? of a regular dodecagon (12 sides) one of whose sides is 9 in.?

10. If the perimeter of a regular decagon is 120, what is the length of one side?

11. Show how to divide a given arc and its central angle into two equal parts; into four equal parts; into eight equal parts.

12. Having divided an arc into a certain number of equal arcs, show how you could divide it into twice that number of equal arcs.

13. Show how a circle may be divided into two equal parts; four equal parts; eight equal parts; sixteen equal parts; 2^n equal parts.

14. If a chord of constant length is laid off successively on a circle, what relation exists between the arcs of these chords? between the central angles of these arcs?

15. What relation exists between the angles formed by the successive chords in Ex. 14? Explain your answer.

16. How many degrees in each angle of a regular polygon of 3 sides? 4 sides? 5 sides? 6 sides? 8 sides? 10 sides?

503. Dividing a circle into equal parts. We may think of a circle as being divided into any number of equal arcs, but the problem of actually dividing a circle into a certain number of equal arcs by means of ruler and compasses can be solved only for certain cases which we shall now consider.

504. *Construction problem.* *To inscribe a square in a given circle, and to circumscribe a square about a given circle.*

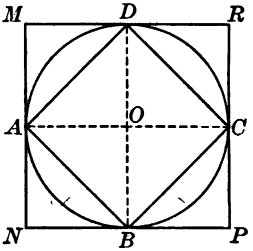


Fig. 481

Given the circle *O*.

Required *to inscribe a square in and circumscribe a square about O.*

Construction 1. Draw two perpendicular diameters AC and BD and connect A, B, C, and D.

Then $ABCD$ is the required inscribed square.

Construction 2. Draw tangents at the points A, B, C, and D.

Then $MNPR$ is the required circumscribed square.

Proof. The proof is left to the student.

505. *Corollary 1.* *By bisecting the arcs subtended by an inscribed square, a regular polygon of 8 sides may be inscribed in a circle; by continuing the process regular polygons of 16, 32, 64, · · · sides may be inscribed.*

506. *Corollary 2.* *Regular polygons of 8, 16, 32, 64, · · · sides may be circumscribed about a given circle.*

SUGGESTION. Bisect the arcs AB, BC, CD, and DA (Fig. 481), and draw tangents at the points of division.

NOTE. The student can now see from Arts. 504 and 505 that regular inscribed polygons the number of sides of which is a number of the form 2^n, where $n \geqq 2$ and is an integer, can be constructed by means of the ruler and compass only. Thus, when $n = 2$, $2^n = 4$; when $n = 3$, $2^n = 8$; and so on indefinitely.

Carl Friedrich Gauss (1777–1855) was the greatest German mathematician of a hundred years ago; indeed, he was one of the two or three greatest mathematicians then living. Even as a child he showed remarkable powers in computing, and as a man he worked in all lines of mathematics and physics. He was one of the inventors of the electric telegraph and was also one of the leading astronomers of his time

507. Theorem. *If a circle is divided into any number of equal arcs, the chords joining the successive points of division form a regular inscribed polygon of the same number of sides.*

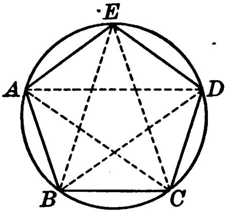


FIG. 482

Given the circle *ABC* . . . divided into the equal arcs *AB*, *BC*, *CD*, *DE*, and *EA*.

***To prove that** ABCDE is a regular inscribed polygon.*

Proof. The proof is left to the student.

508. Corollary 1. *If from the midpoint of each arc cut off by a side of a regular inscribed polygon lines be drawn to the extremities of the arc, a regular inscribed polygon of double the number of sides is formed.*

509. Corollary 2. *An equilateral polygon inscribed in a circle is regular.*

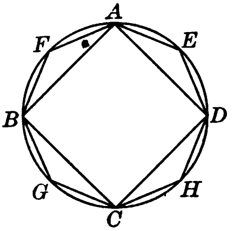


FIG. 483

HISTORICAL NOTE. The general problem of constructing a regular polygon has interested the race for a long time, although this division of the circle cannot be made in all cases by the methods of elementary geometry. It was known in Euclid's time that regular polygons of 2^n, $3 \cdot 2^n$, $5 \cdot 2^n$, and $15 \cdot 2^n$ sides could be constructed.

It remained for Gauss in 1801 to show that the regular polygon of 17 sides can be constructed with ruler and compasses. He showed that in general it is possible to construct with ruler and compasses any polygon if the number of its sides is prime and of the form $2^n + 1$.

510. *Theorem.* *If a circle is divided into any number of equal arcs, the tangents drawn at the points of division form a regular circumscribed polygon of the same number of sides.*

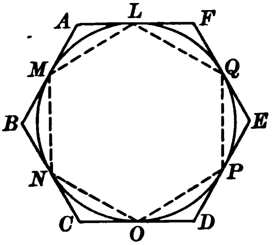


FIG. 484

Given the circle *ABCD* $\cdots$, with the equal arcs *LM*, *MN*, *NO*, $\cdots$, and the lines *AF*, *AB*, $\cdots$ drawn tangent to the circle at *L*, *M*, $\cdots$

To prove that *ABC $\cdots$ is a regular circumscribed polygon of n sides.*

Proof. The proof is left to the student.

SUGGESTION. Draw $ML, MN, \cdots$, and prove that $\triangle$s $AML, MBN, \cdots$ are congruent isosceles triangles. Then $\angle A = \angle B = \cdots$, $AM = AL$, $MB = BN$, $\cdots$, and $AM = MB = BN = \cdots$.

511. *Corollary 1.* *Tangents drawn at the midpoints of the arcs into which a circle is divided by the points of contact of the sides of a regular circumscribed polygon form with those sides a regular polygon of twice as many sides.*

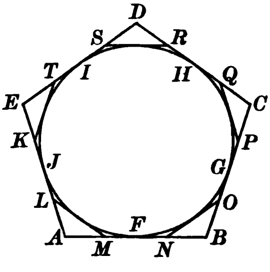


FIG. 485

512. *Corollary 2.* *If tangents are drawn at the vertices of a regular inscribed polygon, they form a regular circumscribed polygon of the same number of sides.*

EXERCISES

1. An equiangular circumscribed polygon is regular.

2. An equiangular inscribed polygon is regular if the number of sides is odd.

3. An equilateral circumscribed polygon is regular if the number of sides is odd.

4. Tangents drawn at the midpoints of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.

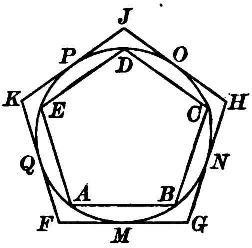


FIG. 486

5. Show that Corollaries 504, 505, and 506 may be stated as follows: A regular polygon of 2^n sides may be inscribed in or circumscribed about a given circle, where n is any positive integer or zero.

513. Formulas relating to the inscribed and circumscribed square. The following exercises develop formulas which relate to an inscribed or circumscribed square.

EXERCISES

1. Prove that a side of an inscribed square is equal to the radius of the circle multiplied by $\sqrt{2}$; that is, show that $s = r\sqrt{2}$.

2. Prove that the side of a circumscribed square is equal to twice the radius of the circle.

3. In terms of the diameter of the circle write the formula for a side of an inscribed square; of a circumscribed square.

4. Write formulas for the perimeters of the inscribed and circumscribed squares in terms of the radii of the respective circles; in terms of the diameters of the circles.

5. Write formulas for the areas of the inscribed and circumscribed squares in terms of the radii of the respective circles; in terms of the diameters of the circles.

6. The area of a square is 36 sq. in. Find the radii of the inscribed and circumscribed circles.

7. The radii of two circles are 4 in. and 6 in. respectively. Find the ratio of the areas of the inscribed squares; of the circumscribed squares.

8. Compare the ratio of the areas of the squares in Ex. 7 with the ratio of the squares of the radii.

9. If the area of a square is 900 sq. in., find the apothem and the radius.

514. Construction problem. *To inscribe a regular hexagon in a given circle.*

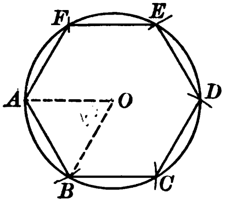


FIG. 487

Given the circle O.

***Required** to inscribe a regular hexagon in the $\odot O$.*

Construction. Draw the radius OA, and with A as center and OA as radius draw an arc cutting the circle at B.

Then AB is a side of the required hexagon.

Proof. The proof is left to the student, who should draw OB and show that $\widehat{AB}$ subtends a central angle of 60°.

515. Corollary 1. *To inscribe a regular triangle in a given circle.*

516. Corollary 2. *To inscribe regular polygons of 12, 24, 48, · · ·, $3 \cdot 2^n$ sides in a given circle, where n is any positive integer or zero.*

517. Corollary 3. *To circumscribe a regular hexagon about a given circle.*

518. ***Corollary 4.*** *Regular polygons of 12, 24, 48, · · ·, $3 \cdot 2^n$ sides may be circumscribed about a given circle, where n is any positive integer or zero.*

EXERCISES

1. The side of a regular hexagon is equal to the radius of the circumscribed circle.

2. A side of a regular inscribed triangle is equal to the radius of the circle multiplied by $\sqrt{3}$.

SUGGESTION. Let x (Fig. 488) represent a side of the regular inscribed triangle. Then show that the $\triangle ADC$ is a right triangle.

The tangent ratio may also be used.

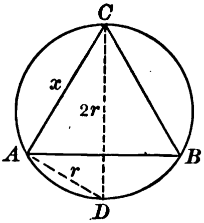


FIG. 488

3. What is the locus of the midpoints of the sides of all regular hexagons which are inscribed in a given circle?

4. Inscribe a circle in a given regular hexagon.

5. Show that a side of a circumscribed equilateral triangle is $2r\sqrt{3}$ and that the area is $3r^2\sqrt{3}$.

SUGGESTION. Use the Pythagorean theorem or the tangent ratio.

6. Prove that the area of an inscribed equilateral triangle is $\frac{3r^2\sqrt{3}}{4}$.

7. In a given circle find the ratio of the areas of the inscribed and circumscribed equilateral triangles.

8. Find the apothem of an inscribed equilateral triangle.

9. Prove that a side of a circumscribed regular hexagon is $\frac{2r}{3}\sqrt{3}$ and that the area is $2r^2\sqrt{3}$.

10. Find the area of a regular hexagon whose side is 8 in.

11. The radius of a circle is 6. Find the area of the inscribed regular hexagon.

12. If the area of a circle is $24\sqrt{3}$, what is the radius?

13. Show that the side of a regular inscribed octagon is $r\sqrt{2-\sqrt{2}}$, that its apothem is $\frac{r}{2}\sqrt{2-\sqrt{2}}$, and that its area is $2r^2\sqrt{2}$.

14. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

519. Extreme and mean ratio. In order to inscribe a regular decagon in a circle it is necessary to know how to divide a line segment in extreme and mean ratio. A line segment AB (Fig. 489) is said to be divided in *extreme and mean ratio* by a point P when the longer part AP is a mean proportional between the whole segment AB and the shorter part PB. Thus, AB is divided internally in extreme and mean ratio at P if

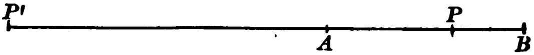


FIG. 489

$$\frac{AB}{AP}=\frac{AP}{PB};$$

AB is divided externally in extreme and mean ratio at P' if

$$\frac{AB}{AP'}=\frac{AP'}{PB}.$$

NOTE. Since this second type of division is not used in elementary geometry, we shall make no application of it in this book.

EXERCISES

1. In Fig. 489 let $AP=x$ and $AB=a$. Show that $x=\frac{a}{2}(\sqrt{5}-1)$ $=0.618\,a$ (approximately) and that $PB=\frac{a}{2}(3-\sqrt{5})$.

2. In Fig. 489, if $AB=8$, find AP.

NOTE. The analysis used in solving Ex. 1 above is algebraic analysis in contrast to geometric analysis. Frequently, however, a purely geometric proof may be found.

520. *Construction problem.* *To divide a line segment in extreme and mean ratio.*

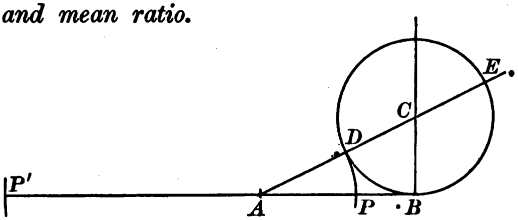


FIG. 490

Given the line segment *AB*.

***Required** to divide AB in extreme and mean ratio; that is, to find a point P such that*

$$\frac{AB}{AP}=\frac{AP}{PB}.$$

Construction. At B erect $CB \perp AB$ and equal to $\frac{1}{2}AB$. With C as center and with radius CB draw a circle. Draw AC, extending it to E. With A as center and the radii AD and AE describe two arcs cutting AB at P and BA produced at P' respectively.

Then AB is divided in extreme and mean ratio as required.

Proof

STATEMENTS	REASONS
1. We shall first consider the proof for the point P.	1. By agreement.
2. $\frac{AE}{AB}=\frac{AB}{AD}$.	2. If a tangent and a secant to a circle are drawn from a point outside the circle, the tangent is a mean proportional between the secant and its external segment.
3. Then $\overline{AB}^2=AD\cdot AE$.	3. If four quantities are in proportion, the product of the means equals the product of the extremes.
4. $\overline{AB}^2=AP(AD+DE)$.	4. Substituting AP for AD, and $AD+DE$ for AE, in 3.
5. Then $\overline{AB}^2=AP(AP+AB)$.	5. Substituting AP for AD, and AB for DE, in 4.

Proof (*Continued*)

STATEMENTS	REASONS
6. $\overline{AB}^2 = \overline{AP}^2 + AP \cdot AB$.	6. Multiplying in 5.
7. $\therefore \overline{AB}^2 - AP \cdot AB = \overline{AP}^2$, or $AB(AB - AP) = \overline{AP}^2$.	7. By subtracting $AP \cdot AB$ from both sides of 6 and factoring.
8. $\therefore AB \cdot PB = \overline{AP}^2$.	8. $PB = AB - AP$.
9. $\therefore \frac{AB}{AP} = \frac{AP}{PB}$.	9. If the product of two factors is equal to the product of two other factors, either pair may be made the means and the other pair the extremes of a proportion.

NOTE. The student should supply the proof for the point P', observing that if directed line segments are brought into the discussion, then $AP = -PA$. Then the proofs for the two points are identical if D and E are interchanged and P' is put in place of P.

HISTORICAL NOTE. The preceding problem is known historically as the "Golden Section," a problem which held great interest to the early Greek geometers. It represented to them the most artistic division of a line segment into two parts. Several theorems in geometry are concerned with this problem; for example, the one in Art. 521. The Golden Section is also used in architectural designs and is the basis for designs in other forms of art and for the metrical relations in nature.

521. *Construction problem.* *To inscribe a regular decagon in a given circle.*

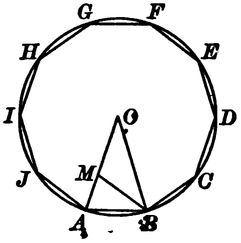


FIG. 491

Given the circle whose center is *O*.

Required *to inscribe a decagon.*

Construction. Divide the radius OA in extreme and mean ratio at M, making $\frac{OA}{OM} = \frac{OM}{MA}$. With A as center and OM as radius strike an arc at B. Then with B as a center and the same radius strike an arc at C, and so on all around the circle. Draw AB, BC, CD, $\cdots$, JA.

The polygon $ABCD \cdots J$ is the required decagon.

Proof

STATEMENTS	REASONS
1. Draw OB and MB.	1. One and only one straight line can be drawn between two points.
2. $\frac{OA}{OM} = \frac{OM}{MA}$, and $OM = AB$.	2. By construction.
3. Then $\frac{OA}{AB} = \frac{AB}{MA}$.	3. By substitution.
4. Also, $\angle A = \angle A$.	4. Identical.
5. $\therefore \triangle AOB \sim \triangle MBA$.	5. If the ratio of two sides of one triangle is equal to the ratio between the two homologous sides of another triangle, and the angles included between these sides are equal, the triangles are similar.
6. Then $\frac{OA}{OB} = \frac{BA}{MB}$.	6. Homologous sides of similar triangles are proportional.
7. But $OA = OB$.	7. Radii of the same circle are equal.
8. $\therefore \frac{BA}{MB} = 1$.	8. Obvious.
9. $\therefore MB = BA$.	9. Obvious.
10. $\therefore \angle MAB = \angle AMB$.	10. The base angles of an isosceles triangle are equal.
11. Also, $MO = MB$.	11. Things equal to the same thing are equal to each other.
12. $\therefore \angle MOB = \angle MBO$.	12. The base angles of an isosceles triangle are equal.

Proof (*Continued*)

STATEMENTS	REASONS
13. But $\angle AMB = \angle MOB + \angle MBO$.	13. The exterior angle of any triangle is equal to the sum of the two nonadjacent interior angles of the triangle.
14. $\therefore \angle AMB = 2\angle MOB$.	14. By substitution.
15. And $\angle MAB = 2\angle MOB$.	15. Because $\angle MAB = \angle AMB$.
16. And $\angle OBA = 2\angle MOB$.	16. Because $\angle OBA = \angle MAB$.
17. But $\angle MAB + \angle ABO + \angle MOB = 180°$.	17. The sum of the interior angles of a triangle is 180°.
18. $\therefore 5\angle MOB = 180°$.	18. By substitution.
19. Then $\angle MOB = 36°$.	19. Equals divided by equals give equals.
20. $\therefore \widehat{AB} = \frac{1}{10}$ of a circle.	20. Obvious.
21. $\therefore ABCD \cdots J$ is a regular inscribed decagon.	21. If a circle is divided into any number of equal arcs, and the points of division are joined by chords, the figure thus formed is a regular inscribed polygon.

522. **Corollary 1.** *To inscribe a regular pentagon in a circle.*

523. **Corollary 2.** *To inscribe regular polygons of 5, 10, 20, 40, 80, $\cdots$, $5 \cdot 2^n$ sides in a given circle, where n is any positive integer or zero.*

524. **Corollary 3.** *To inscribe a regular polygon of fifteen sides (pentadecagon) in a circle.*

SUGGESTION. Find the difference between the arc of the side of a regular inscribed hexagon (60°) and the arc of the side of a regular inscribed decagon (36°). The chord of this arc (24°) is a side of the required regular inscribed polygon of fifteen sides (see Fig. 492).

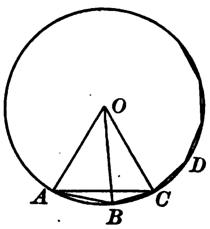


FIG. 492

525. **Corollary 4.** *By drawing tangents at the vertices of regular inscribed polygons of 5, 10, 20, $\cdots$, $5 \cdot 2^n$ sides, regular polygons of the same number of sides may be circumscribed about a circle.*

EXERCISES

1. Regular polygons of 15, 30, 60, $\cdots$, $15 \cdot 2^n$ may be inscribed in a circle.

2. By drawing tangents at the vertices of regular inscribed polygons of 15, 30, 60, $\cdots$, $15 \cdot 2^n$ sides, regular polygons of the same number of sides may be circumscribed about a circle.

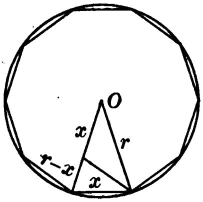


FIG. 493

3. Show that the side of a regular inscribed decagon is $\frac{r}{2}(\sqrt{5}-1)$.

SUGGESTION. Let x represent a side of the decagon (Fig. 493). Then, by the extreme and mean ratio construction of Art. 520, $\frac{r}{x} = \frac{x}{r-x}$. Solve this equation for x.

4. Show that the apothem of a regular inscribed decagon is $\frac{r}{4}\sqrt{10+2\sqrt{5}}$.

5. Show that the area of a regular inscribed decagon is $\frac{5}{4} r^2 \sqrt{10-2\sqrt{5}}$.

6. Show that the side of a regular inscribed pentagon is $\frac{r}{2}\sqrt{10-2\sqrt{5}}$.

SUGGESTION. Let x (Fig. 494) represent one half the side AC of the pentagon and let y represent the side BC of the decagon.

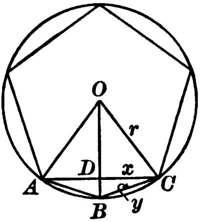


FIG. 494

Then $\overline{OD}^2 = r^2 - x^2$,

and $OD = \sqrt{r^2 - x^2}$.

$\therefore DB = r - \sqrt{r^2 - x^2}$.

Then $\overline{DC}^2 = \overline{BC}^2 - \overline{DB}^2$.

$\therefore x^2 = y^2 - \left(r - \sqrt{r^2 - x^2}\right)^2$.

By Ex. 3, $y^2 = \frac{r^2}{4}(\sqrt{5}-1)^2$.

$$\therefore x^2 = \frac{r^2}{4}(\sqrt{5}-1)^2 - \left(r - \sqrt{r^2 - x^2}\right)^2.$$

Solving the above equation for x, we have

$$x = \frac{r}{4}\sqrt{10 - 2\sqrt{5}},$$

and
$$2x = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}.$$

The student should write out the complete proof.

7. Show that the value of $\frac{r}{2}(\sqrt{5} - 1)$ is approximately $0.618\,r$ and that the value of $\frac{r}{2}\sqrt{10 - 2\sqrt{5}}$ is approximately $1.1755\,r$.

8. By a trigonometric method find the side of a regular inscribed pentagon and the side of a regular inscribed decagon.

SUGGESTION. Use the sine ratio.

9. The construction shown in Fig. 495 gives a practical method of constructing the sides of a regular decagon and pentagon. The line DO is drawn $\perp AC$, and OC is bisected at E. With E as center and ED as radius $\overset{\frown}{DF}$ is drawn. Then the chord DF is drawn. The sides FD, DO, and FO of the $\triangle OFD$ are equal to the sides of a regular pentagon, hexagon, and decagon respectively. Prove that this is correct.

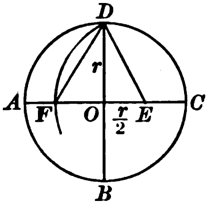


FIG. 495

10. A circular piece of wood has a diameter of 6 ft. What is the side of the largest regular pentagon that can be cut from the piece?

11. Find the side of a regular inscribed dodecagon (a polygon of twelve sides).

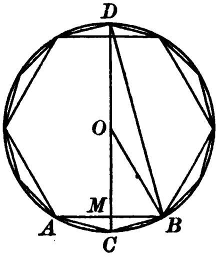


FIG. 496

SUGGESTION. Let AB (Fig. 496) be a side of an inscribed hexagon.

12. Draw a regular hexagon, and draw all its diagonals. How many equilateral triangles are thus made? How many right triangles? How many isosceles triangles?

526. *Theorem.* *A circle can be circumscribed about any regular polygon.*

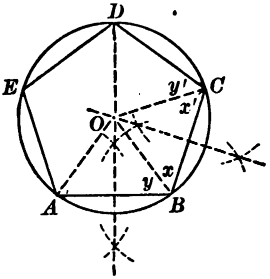


FIG. 497

Given the regular polygon *ABCDE*.

***To prove that** a circle can be circumscribed about ABCDE.*

Construction. Construct the ⊙ O through A, B, and C by the method of Art. 200.

Then the ⊙ O is the required circle.

Proof

STATEMENTS	REASONS
1. In order to prove that the circle O is the required circle, we must show that it passes through D and E.	1. Obvious.
2. Draw OA, OB, OC, OD.	2. One and only one line can be drawn between two points.
3. In the $\triangle$ AOB and COD, $AB = CD$.	3. Given.
4. $OB = OC$.	4. By construction.
5. $\angle ABC = \angle BCD$.	5. By definition of a regular polygon.
6. $\angle x = \angle x'$.	6. The base angles of an isosceles triangle are equal.
7. $\therefore \angle y = \angle y'$.	7. Equals subtracted from equals give equals.

Proof (*Continued*)

STATEMENTS	REASONS
8. $\therefore \triangle AOB \cong \triangle COD$.	8. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.
9. $\therefore OA = OD$.	9. Homologous parts of congruent triangles are equal.
10. $\therefore$ the circle O passes through D.	10. Because D is as far from O as is A and hence lies on the circle by the definition of a circle.
11. In like manner, it may be proved that the circle passes through E.	11. As proved in 2–10 above.
12. $\therefore$ a circle can be circumscribed about any regular polygon.	

527. *Theorem.* *A circle can be inscribed in any regular polygon.*

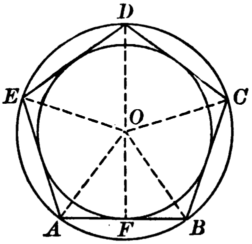


FIG. 498

Given the regular polygon *ABCDE*.

To prove that *a circle can be inscribed in $ABCDE$.*

Construction. Find the center O of the circumscribed circle, and draw $OF \perp AB$. Then with O as center and OF as radius construct the $\odot ABC \cdots$, which is the required circle.

Proof

STATEMENTS	REASONS
1. Draw the circumscribed $\odot ABC \cdots$.	1. This can be done by the previous theorem.
2. Then $AB, BC, \cdots$ are equal chords of the same circle.	2. They are equal by hypothesis.
3. $\therefore$ these chords are equally distant from the center O.	3. In the same circle or in congruent circles equal chords are equidistant from the center.
4. Hence the inner circle passes through the midpoints of each side of the regular polygon.	4. Obvious.
5. $\therefore BC$ is tangent to the inner circle.	5. By definition of a tangent.
6. In like manner, $CD, DE, \cdots$ are tangent to the inner circle.	6. Obvious.
7. $\therefore$ the inner circle is inscribed in the polygon.	7. By definition of an inscribed circle.

528. Corollary 1. *In a given regular polygon the inscribed and circumscribed circles are concentric.*

529. Corollary 2. *The radius to any vertex of a regular inscribed or circumscribed polygon bisects the angle at the vertex.*

530. Corollary 3. *The angles formed by the radii at the center of a regular inscribed or circumscribed polygon are equal, and each is supplementary to an angle of the polygon.*

531. Theorem. *The perimeter of a regular inscribed polygon of $2n$ sides is greater than the perimeter of a regular n-sided polygon inscribed in the same circle.*

Proof. The proof is left to the student.

532. Theorem. *The perimeter of a regular circumscribed polygon of $2n$ sides is less than the perimeter of the regular n-sided polygon circumscribed about the same circle.*

Proof. The proof is left to the student.

NOTE. From the theorems of Arts. 531 and 532 it is evident that

1. The perimeter of a regular inscribed polygon *increases* as the number of sides *increases.*

2. The perimeter of a regular circumscribed polygon *decreases* as the number of sides *increases.*

533. Theorem. *Two regular polygons of the same number of sides are similar.*

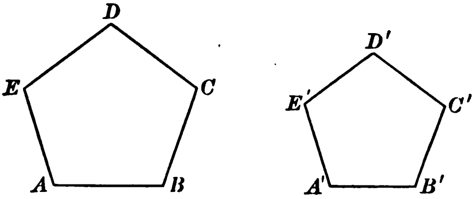


FIG. 499

Given the regular polygons *ABCDE* and *A'B'C'D'E'* with the same number of sides.

To prove that $ABCDE \sim A'B'C'D'E'$.

Proof

STATEMENTS	REASONS
1. The two polygons are mutually equiangular.	1. By definition of a regular polygon each angle is 108°.
2. $AB = BC = CD = \cdots$, and $A'B' = B'C' = C'D' = \cdots$.	2. By definition of a regular polygon.
3. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \cdots$.	3. If equals are divided by equals, the quotients are equal.
4. ∴ the homologous sides of the two polygons are proportional.	4. This is obvious from 3.
5. $\therefore ABCDE \sim A'B'C'D'E'$.	5. By definition of similar polygons.

534. Corollary. *The areas of two regular polygons of the same number of sides are to each other as the squares of their sides.*

535. *Theorem.* *The perimeter of two regular polygons of the same number of sides are to each other as their radii or as their apothems.*

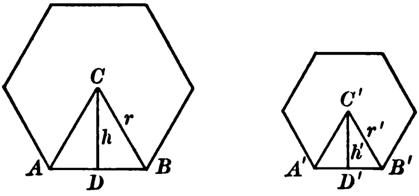


FIG. 500

Given the regular polygons $AB\cdots$ and $A'B'\cdots$ of the same number of sides, with perimeters P and P', radii r and r', and apothems h and h' respectively.

To prove that $\dfrac{P}{P'}=\dfrac{r}{r'}=\dfrac{h}{h'}$.

Proof

STATEMENTS	REASONS
1. Let C and C' be the respective centers of the polygons $AB\cdots$ and $A'B'\cdots$.	1. Obvious.
2. Draw the radii CA, CB, $C'A'$, and $C'B'$.	2. One and only one line can be drawn between two points.
3. Draw the apothems h and h'.	3. From a point outside a given line one and only one perpendicular can be drawn to the line.
4. The polygons $AB\cdots$ and $A'B'\cdots$ are similar.	4. Regular polygons of the same number of sides are similar.
5. $\therefore \dfrac{P}{P'}=\dfrac{AB}{A'B'}$.	5. The perimeters of two similar polygons have the same ratio as any two homologous sides.
6. $\triangle CAB \sim \triangle C'A'B'$.	6. Two similar polygons can be divided into the same number of triangles, similar each to each and similarly placed.

Proof (*Continued*)

STATEMENTS	REASONS
7. $\therefore \frac{AB}{A'B'} = \frac{r}{r'}$.	7. The homologous sides of similar triangles are proportional.
8. Also, $\frac{AB}{A'B'} = \frac{h}{h'}$.	8. The sides of similar triangles are to each other as their altitudes.
9. $\therefore \frac{P}{P'} = \frac{r}{r'} = \frac{h}{h'}$.	9. Things equal to the same thing are equal to each other.

536. Corollary. *The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.*

537. Circumference. The length of a circle in linear units is called the *circumference.* From the time of the early geometers until recently there have been many attempts to find, by geometric methods, a straight line which is equal in length to a circle of a given radius. We know now, however, that by any method used in elementary geometry we cannot express the circumference of a circle in the way in which we have defined the length of a straight-line segment; that is, as the ratio of the given segment to another line segment which is taken as the unit of measure. We shall therefore have to use an entirely different method to find the length of the circle in linear units.

The following exercises illustrate a method which is neither accurate nor geometric but which will serve as a preparation for what is to follow.

EXERCISES

1. On cardboard draw a circle which is 6 in. in diameter. Cut out the circle and place a thread or string around the edge so as to touch the circle at all points. Measure the length of the string and find the ratio of this length to the diameter.

2. Carry out the instructions of Ex. 1 for circles which have diameters of 3, 4, 7, 8, and 10 in. respectively. Tabulate the results as follows:

DIAMETER (D)	LENGTH OF THREAD (C)	RATIO $\frac{C}{D}$
3		
4		
6		
7		
8		
10		

3. In Ex. 2 what seems to be the most probable value of the ratio $\frac{C}{D}$?

4. Verify your answer to Ex. 3 by measuring other circular objects in the classroom.

538. Length of a circle. In the theorems of Arts. 531 and 532 we have seen that the perimeter of a regular inscribed polygon increases as the number of sides increases, and that the perimeter of a regular circumscribed polygon decreases as the number of sides increases.

We shall here assume that the perimeters of successive inscribed and circumscribed polygons approach nearer and nearer to each other as the number of their sides is continually increased.

It can be shown that the length of a circle is greater than the perimeter of any inscribed polygon, and we shall also assume that the length of the circle is less than the perimeter of any circumscribed polygon. Obviously, then, the length of a circle lies between the lengths of the perimeters of any pair of inscribed and circumscribed polygons. We can compute these perimeters for any pair of given inscribed and circumscribed polygons and thus get the circumference of the circle to any required degree of approximation. It is fairly easy to compute the perimeters of pairs of successive inscribed and circumscribed polygons each pair of which has twice as many sides as the preceding pair.

539. Finding the perimeter of a regular inscribed polygon. Let AB (Fig. 501) be the side of a regular inscribed n-gon, and draw $OC \perp AB$.

Then $$\angle BOC = \left(\frac{360}{2n}\right)^\circ. \qquad \text{Why?}$$

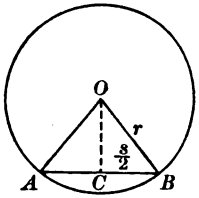


FIG. 501

If s represents AB and r represents the radius of the circumscribed circle, then

$$\sin\left(\frac{360}{2n}\right)^\circ = \frac{\frac{s}{2}}{r},$$

and $$s = \sin\left(\frac{360}{2n}\right)^\circ d. \text{ Why?}$$

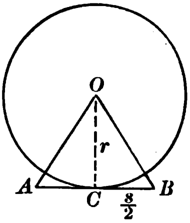


FIG. 502

Then p, the entire perimeter, is given by the formula

$$p = n\left[\sin\left(\frac{360}{2n}\right)^\circ\right]d. \qquad \text{Why?}$$

This formula, together with the one given in the following article, will enable us to compute the perimeters of inscribed and circumscribed regular polygons and thus get an approximate value for the length of a circle.

540. Finding the perimeter of a regular circumscribed polygon. Let AB (Fig. 502) be the side of a regular circumscribed n-gon, and draw $OC \perp AB$.

Then $$\angle BOC = \left(\frac{360}{2n}\right)^\circ. \qquad \text{Why?}$$

If S represents AB and r represents the radius of the inscribed circle, then

$$\tan\left(\frac{360}{2n}\right)^\circ = \frac{\frac{S}{2}}{r},$$

and

$$S = \tan\left(\frac{360}{2n}\right)^\circ d. \qquad \text{Why?}$$

Hence P, the entire perimeter, is given by the formula

$$P = n\left[\tan\left(\frac{360}{2n}\right)^\circ\right]d. \qquad \text{Why?}$$

541. Finding the length of a circle. By using the formulas of Arts. 539 and 540 the perimeters of the inscribed and circumscribed polygons shown in the following table may be obtained. The student should check a few of the results to see how the formulas are used.

NUMBER OF SIDES	PERIMETER OF THE INSCRIBED POLYGON	LENGTH OF THE CIRCLE (l)	PERIMETER OF THE CIRCUMSCRIBED POLYGON
3	$2.5980 \cdots d$	$2.5d < l < 5.2d$	$5.1963 \cdots d$
4	$2.8284 \cdots d$	$2.8d < l < 4.0d$	$4.0000 \cdots d$
5	$2.9390 \cdots d$	$2.9d < l < 3.7d$	$3.6325 \cdots d$
6	$3.0000 \cdots d$	$3.0d < l < 3.5d$	$3.4644 \cdots d$
8	$3.0614 \cdots d$	$3.0d < l < 3.4d$	$3.3137 \cdots d$
12	$3.1058 \cdots d$	$3.1d < l < 3.3d$	$3.2153 \cdots d$
16	$3.1215 \cdots d$	$l = 3.1d$ (approximately)	$3.1826 \cdots d$
90	$3.1410 \cdots d$	$l = 3.14d$ (approximately)	$3.1427 \cdots d$

The preceding table, although not carried very far, shows how closely the numbers which represent the perimeters of the inscribed and circumscribed polygons agree as the number of sides of the polygon is increased.

The student is not expected to carry the table further, but we shall give below a table which shows the value of the perimeter (p) of the inscribed polygons and that of the perimeter (P) of the circumscribed polygons to five decimal places for a number of polygons in which the diameter of the circle is taken as 1. In each case in the following table the subscript indicates the number of sides of the polygons:

$$\begin{array}{rr}
p_4 = 2.82843 & P_4 = 4.00000 \\
p_8 = 3.06147 & P_8 = 3.31371 \\
p_{16} = 3.12145 & P_{16} = 3.18260 \\
p_{32} = 3.13655 & P_{32} = 3.15172 \\
p_{64} = 3.14033 & P_{64} = 3.14412 \\
p_{128} = 3.14128 & P_{128} = 3.14222 \\
p_{256} = 3.14151 & P_{256} = 3.14175 \\
p_{512} = 3.14157 & P_{512} = 3.14163
\end{array}$$

The last two perimeters in the preceding table agree to three decimal places, and if we stop here we may say that the length of the circle is $3.141 \cdots d$. It is clear, however, that a closer approximation can be obtained by considering the perimeters of inscribed and circumscribed polygons with still greater numbers of sides. The perimeters in each pair approach each other closer and closer; but no matter how close the perimeters of any chosen pair of inscribed and circumscribed polygons approach each other, there is always another pair whose perimeters approach still closer to the length of the circle. For example, it is found that

$$p_{8192} = 3.1415928 \cdots d,$$

and

$$P_{8192} = 3.1415926 \cdots d.$$

If we take the length of the circle as $3.141592 \cdots d$, the result is a very close approximation.

In general, although we cannot get an exact value for the constant multiplier, it is evident that the length (C) of a circle is a multiple of the diameter. This constant multiplier is represented by the Greek letter π (pi), and the value usually assigned to it is 3.1416. In practical computation $\frac{22}{7}$ is often used as the value of π.

Finally, we see that the formula which gives the length of a circle in terms of its diameter is

$$C = \pi d.$$

542. ***Corollary 1.*** *The circumference of a circle equals* $2\pi r$, *where* r *is the number of linear units in the radius.*

543. ***Corollary 2.*** *The circumferences of two circles have the same ratio as their diameters or as their radii.*

EXERCISES

1. Find the circumference of a circle whose diameter is 3 cm.; 5 cm.; 10 cm.

2. What length of strip iron is required for the tire of a wagon wheel $3\frac{1}{2}$ ft. in diameter?

3. In rolling 50 yd. how many revolutions are made by a wheel whose radius is 4 ft.?

4. Find the width of the ring between two concentric circles whose lengths are 2.5 ft. and 3.5 ft. respectively.

5. Disregarding any overlapping, how wide should a strip of sheet iron be cut in order to make a stovepipe 10 in. in diameter?

6. Taking the radius of the earth as 4000 mi., what is the circumference of the earth at the equator?

7. In a circle whose radius is 4 ft. what is the length of the arc which is cut off by a central angle of 60°?

8. A circular garden has a diameter of 40 ft. At $3 per square yard what is the cost of constructing a cement walk 3 ft. wide around this garden?

544. Theorem. *The area of a regular polygon is equal to one half the product of its apothem and its perimeter.*

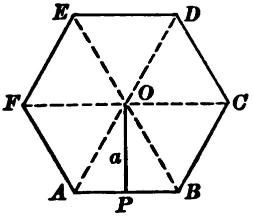


FIG. 503

Given the regular polygon *ABCDE*.

***To prove that** the area of ABCDE equals one half the product of its apothem and its perimeter.*

Proof

STATEMENTS	REASONS
1. From O, the center of the polygon, draw the radii OA, OB, $OC, \cdots$, and draw the apothem a. Each of the triangles thus formed has an altitude equal to the apothem a.	1. Obvious.
2. The area of $\triangle AOB = \frac{a}{2}AB.$	2. The area of a triangle is equal to one half the product of its altitude and its base.
3. In like manner, the area of $\triangle BOC = \frac{a}{2}BC,$ of $\triangle COD = \frac{a}{2}CD,$ and so on.	3. Same reason.
4. $\therefore$ the area of the polygon equals one half the product of the perimeter and the apothem.	4. Since the whole is equal to the sum of its parts, we add the areas of the triangles.

545. Sector of a circle. The part of a plane bounded by two radii of a circle and their intercepted arc is called a sector of the circle (Art. 297).

The student should observe that the sector shown in Fig. 504 differs from a triangle only in that the base of a triangle is a straight line, while the corresponding line for the sector is an arc. But as the length of the arc of a sector is decreased, it is correspondingly more difficult to tell whether the base line is straight or curved. Hence for all practical purposes *the area of a sector of a circle is equal to one half the product of the radius and the length of the arc of the sector.* It can be proved that this is true, but the proof is difficult. Accordingly, in this course we may accept the statement without proof.

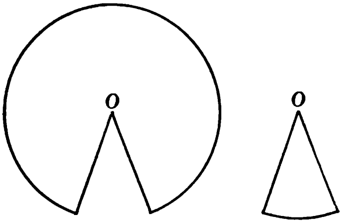


FIG. 504

546. Finding the area of a circle. When we learned how to find the area of a parallelogram (Art. 448), we saw that it depended upon the area of the rectangle, which we had already found (Art. 441) by dividing a rectangle into unit squares and then counting the squares. We also saw that the area of a triangle (Art. 454) depended upon the area of the parallelogram, and that the area of a trapezoid (Art. 456), in turn, depended upon the area of the triangle.

When we come to the problem of finding the area of a circle, we encounter a difficulty; namely, that the part of the plane whose area we wish to measure is bounded by a circle instead of by intersecting straight lines. That is to say, we cannot divide the figure into squares or rectangles as we did before and as we might like to do here. We can, however, divide the part of the plane bounded by the circle into equal sectors, as shown in Fig. 505.

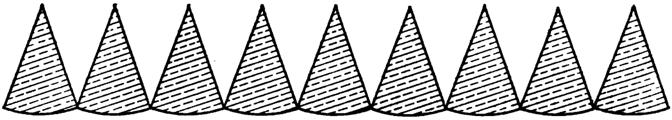


FIG. 505

547. Theorem. *The area of a circle is equal to one half the product of its radius and its circumference.*

SUGGESTION. Divide the circle into several sectors and find the area of the circle by adding the areas of the sectors.

548. Corollary 1. *The area of a circle is equal to* π *times the square of its radius; that is,* $A = \pi r^2$.

549. Corollary 2. *The area of a circle equals* $\frac{\pi d^2}{4}$.

550. Corollary 3. *The areas of two circles are to each other as the squares of their radii or as the squares of their diameters.*

EXERCISES

1. Draw a graph of the relation $A = \pi r^2$, and show why the area of a circle is a function of the radius.

2. In Ex. 1 what happens to A if r is doubled? if r is trebled? if r is quadrupled?

3. By means of the graph of Ex. 1 compare the way in which the area increases for values of r less than 1 with the way in which it increases for values of r greater than 1.

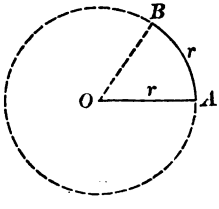


FIG. 506

551. Radian measure. If we think of the length of an arc in terms of the radius of the circle as a unit instead of in degrees, we obtain a new unit of arc measure. This unit is an arc of a circle whose length in linear units is equal to the length of the radius of the circle, and is called a *radian.* In Fig. 506 $\widehat{AB}$, whose length is equal to the radius OA, is a radian.

552. Theorem. *The length of an arc of a circle in linear units is to the circumference as the number of degrees in the arc is to 360.*

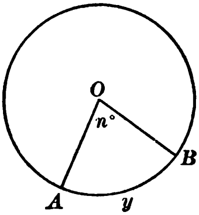


FIG. 507

Given the arc y containing n degrees on the circle O.

To prove that $\dfrac{y}{2\pi r} = \dfrac{n}{360}$.

Proof

STATEMENTS	REASONS
1. The entire angle about the center O is 360°.	1. By definition of a perigon.
2. The angle of 360° is measured by $2\pi r$.	2. An angle at the center of a circle is measured by its arc on the circle.
3. $\therefore \dfrac{y}{2\pi r} = \dfrac{n}{360}$.	3. In the same circle or in congruent circles two central angles have the same ratio as their arcs.

NOTE. Whenever the measure of an angle is expressed in terms of π or any other numeral or numerals and no angular unit is given, the radian is always understood as the unit.

EXERCISES

1. If a circle has a radius of 6 in., what is the length of an arc of 30°?

2. An arc of a circle is 10π inches long and has a central angle of 20°. Find the radius of the circle.

3. If the radius of a circle is 8 in. and an arc is 6π units long, how many degrees are there in the central angle of the arc?

553. Value of the radian. It will now be easy to show the truth of the following important relation: *The sum of the angles about a point in a plane is 2π radians.*

If r is the radius of a circle, the circumference is equal to $2\pi r$. Then, by definition of a radian, the circumference contains 2π radians. This circumference measures the sum of the angles about a point (a perigon), and hence the above relation is true.

Furthermore, we know that a perigon contains 360°, and hence it follows that

$$2\pi \text{ radians} = 360^\circ,$$

or

$$1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}. \qquad (1)$$

But, as has already been shown,

$$\pi \doteq 3.14159265+.$$

$$\therefore 1 \text{ radian} = 57^\circ.2957795, \text{ or } 57.^\circ 3. \qquad (2)$$

From (1) we also can show that

$$1^\circ = \frac{\pi}{180} \text{ radians} = 0.0174533 \text{ radian}. \qquad (3)$$

The preceding equations will be useful to the student in finding the number of degrees in an angle or arc when its measure is given in radians and vice versa, and will also be valuable in later work in science and mathematics.

554. Theorem. *An angle of θ radians at the center of a circle of radius r intercepts on the circle an arc whose length s is given by the formula $s = r\theta$.*

SUGGESTION. In proving the theorem show that

$$\frac{s}{\text{arc of 1 radian}} = \frac{\theta}{\text{1 radian}},$$

or

$$\frac{s}{r} = \frac{\theta}{\text{1 radian}},$$

$$\therefore s = r\theta.$$

EXERCISES

1. An angle at the center of a circle whose radius is 6 ft. intercepts an arc whose length is 4 ft. Find the number of degrees in the angle.

2. An angle of 53° 20′ is at the center of a circle whose radius is 12 ft. Find the length of the arc intercepted by the sides of the central angle.

3. Express in degrees each of the following angles: $\frac{\pi}{2}$; $\frac{\pi}{3}$; $\frac{2\pi}{3}$; 2π; $\frac{3\pi}{2}$; $\frac{5\pi}{2}$; $\frac{3\pi}{4}$; 4π.

4. Express in radians each of the following angles: 30°; 45°; 60°; 90°; 135°; 150°; 270°; 88° 20′; 42° 20′ 30″.

5. Find the complement and supplement of each of the following angles: $\frac{\pi}{2}$; $\frac{\pi}{3}$; $\frac{\pi}{4}$; $\frac{2\pi}{3}$; $\frac{3\pi}{4}$; π.

6. Find the sine, cosine, and tangent of the angles given in Ex. 5.

7. The difference between the two acute angles of a right triangle is $\frac{\pi}{6}$. Find the size of each angle in degrees; in radians.

555. Theorem. *The area of a sector of a circle is to the area of the circle as the number of degrees in the central angle of the sector is to 360.*

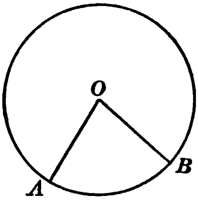


FIG. 508

Given the sector AOB of the circle O.

To prove that $\dfrac{AOB}{\pi r^2}=\dfrac{\angle O}{360}$.

Proof

STATEMENTS	REASONS
1. The area of the circle equals $\dfrac{Cr}{2}$.	1. The area of a circle equals one half the product of the radius and the circumference of the circle.
2. The area of the sector AOB equals $\dfrac{\overset{\frown}{AB}\cdot r}{2}$.	2. The area of a sector equals one half the product of the arc of the sector and the radius of the circle.
3. $\therefore \dfrac{AOB}{\pi r^2}=\dfrac{\overset{\frown}{AB}\cdot\dfrac{r}{2}}{\dfrac{Cr}{2}}$.	3. Equals divided by equals are equal.
4. $\therefore \dfrac{AOB}{\pi r^2}=\dfrac{\overset{\frown}{AB}}{C}$.	4. By simplifying the complex fraction.
5. But $\dfrac{\overset{\frown}{AB}}{C}=\dfrac{\angle O}{360}$.	5. The length of an arc of a circle in linear units is to the circumference as the number of degrees in the arc is to 360.
6. $\therefore \dfrac{AOB}{\pi r^2}=\dfrac{\angle O}{360}$.	6. Things equal to the same thing are equal to each other.

EXERCISE

Practical applications of the preceding theorem can be found in the representation of statistics. Such graphical representations are called *circular pictograms*, as the student will probably remember. Show how the preceding theorem applies in such figures, and discuss the facts brought out by circular pictograms if any are available.

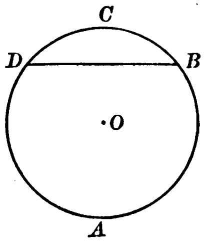


FIG. 509

556. Segment of a circle. The part of a plane inclosed by a chord of a circle and either of the intercepted arcs is called a segment of a circle (Art. 298). Thus, in Fig. 509 the chord DB forms with the arcs DCB and DAB the segments DCB and DAB respectively.

557. Area of a segment. The area of a segment whose arc is less than a semicircle may be found by subtracting the area of the triangle bounded by the two radii and the chord of the segment from the area of the corresponding sector.

Thus, in Fig. 510 the area of

$$\triangle AOB = \frac{x}{2}\sqrt{r^2 - \frac{x^2}{4}},$$

and the area of

$$\text{sector } AOB = \frac{ry}{2},$$

where y is the length of $\overset{\frown}{ACB}$.

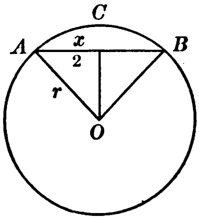


FIG. 510

Therefore the area of the segment BCA is given by the formula

$$A = \frac{yr}{2} - \frac{x}{2}\sqrt{r^2 - \frac{x^2}{4}}.$$

The student should develop this formula.

EXERCISES

1. Show that the area of a segment of a circle is given by the formula $A = \frac{yr}{2} - \frac{r^2}{2} \sin \theta$, where θ is the central angle, y the arc, and r the radius of the circle.

2. How can the area of a segment whose arc is greater than a semicircle be found?

3. Change the following angles into radian measure: 90°; 180°; 270°; 30°; 60°; 20° 15′; 680° 20′; −1120° 4′.

4. Change the following angles whose measures are given in radians into degrees: $\frac{\pi}{2}$; π; $\frac{\pi}{4}$; $\frac{2\pi}{3}$.

5. Prove that the area of a sector of a circle is equal to $\frac{1}{2} r^2\theta$, where r is the radius of the circle and the central angle θ is measured in radians.

6. Prove that the formula $A = \frac{r^2}{2}(\theta - \sin \theta)$ gives the area of a segment of a circle whose arc is equal to θ radians and whose radius is r.

7. Find the area of a segment of a circle when the radius of the circle is 10 ft. and the arc of the segment is 48°.

8. A cow is tied to the middle of one side of a shed 12 ft. square by a rope 18 ft. long. Over what area is the cow free to graze?

9. Using Fig. 511, develop the following formulas for finding the area of a ring:

$$A = \pi (R + r)(R - r),$$

$$A = \frac{\pi x (D + a)}{2}.$$

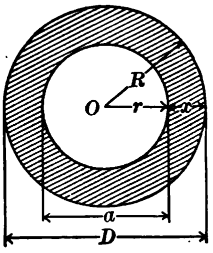


FIG. 511

10. If semicircles are drawn on the sides of a right triangle, prove that the area of the semicircle on the hypotenuse is equal to the sum of the areas of the semicircles on the two legs of the triangle.

11. Semicircles are constructed on the legs of a right triangle as shown in Fig. 512. Show that the sum of the crescents $AECD$ and $CFBG$ is equal to the area of the triangle.

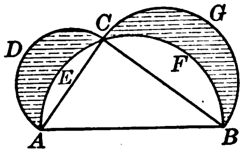


FIG. 512

12. Construct a circle equal to the sum of two given circles.

13. If semicircles are described on the sides of an inscribed square as shown in Fig. 513, show that the sum of the areas of the four crescents which lie outside the circle is equal to the area of the inscribed square.

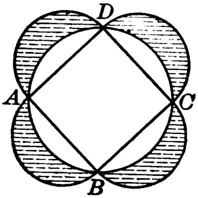


FIG. 513

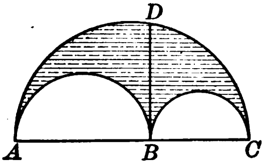


FIG. 514

14. In Fig. 514 semicircles are drawn upon AB, BC, and AC as diameters, and $DB \perp AC$. Prove that the area of the shaded surface bounded by the three arcs AB, BC, and ADC is equal to the area of the circle whose diameter is DB.

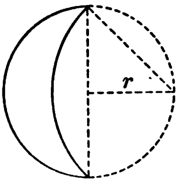


FIG. 515

15. In Fig. 515 prove that the area of the crescent which is bounded by a semicircle and an arc of another circle is equal to r^2, where r is the radius of the semicircle, and the center of the circle containing the arc is on the first circle.

16. The inside diameter of a circular water pipe is 12 ft. Find the area of the internal cross section of the pipe.

17. What fraction of the area of a circle is the area of a sector of that circle whose angle is 15°? 30°? 45°? 36°? 120°?

18. The area of a circle is 144 sq. ft. What is the area of a sector of this circle whose central angle is 30°? whose central angle is 60°?

19. Find the area of a sector of 30° in a circle whose radius is 15.

20. Find the circumference of a circle in which a sector whose central angle is 45° has an area of 15π square feet.

21. Find the area of a segment of a circle whose arc is 90° if the radius of the circle is 9 ft.

22. Three congruent circles each with a radius of 10 in. are tangent externally. Find the area of the part of the plane bounded by the arcs of the three circles.

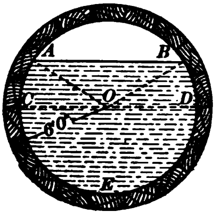


FIG. 516

23. Before the amount of water flowing through a pipe can be determined, the area of the cross section of the water must be known. Find the area of the cross section of the water in Fig. 516 if the water rises to a height of 10 ft. in the pipe.

24. In Fig. 516 find the length of $\widehat{AEB}$ (called the *wetted perimeter* of the pipe) if the water level is AB and $\angle AOB = 45°$. What is the length of the wetted perimeter if $\angle AOB = 60°$? if $\angle AOB = 90°$?

TABLE OF ROOTS AND POWERS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	**55**	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	**65**	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	**75**	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	**85**	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	**90**	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	**95**	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.641

INDEX

Acute angle, 11
Adding integers and fractions, 336
Adding unit fractions, 333
Adjacent angles, 17
Altitude of a triangle, 23
Analysis, method of, 269
Analytic method of proof, 92
Angle, 9
  acute, 11
  bisector of, 23
  central, 14
  complement of, 29
  cosine of, 312
  danger, 234
  of depression, 324
  of elevation, 323
  initial side of, 10
  left side of, 27
  oblique, 11
  obtuse, 11
  reflex, 11
  right, 10
  right side of, 27
  sine of, 311
  straight, 10
  supplement of, 29
  symbol for, 10
  tangent of, 312
  terminal side of, 10
  vertex of, 10
Angles, adjacent, 17
  alternate-exterior, 32
  alternate-interior, 30
Angles, classes of, 10, 11
  complementary, 29
  construction of, 14
  corresponding, 24
  dihedral, 201
  exterior sides of, 17
  geometric addition and subtraction of, 17
  interior on the same side of the transversal, 32
  measurement of, 14
  notation for reading, 12
  opposite, 30
  with parallel sides, 28
  size of, 10
  supplementary, 29
  vertical, 30
Antecedent, 206
Apothem, 402
Approximate measurements and computations, 319
Arc, 14
  major, 212
  minor, 212
Areas, of a circle, 429
  measurement of, 354
  of a parallelogram, 357
  of polygons in general, 359
  practical method of estimating, 354
  proportionality of, 395, 397
  of a rectangle, 355, 356
  of a right triangle, 356
  of a segment of a circle, 435

Areas, of a square, 356
  of a trapezoid, 358
  of a triangle, 358, 388, 390
  unit of, 354
Axioms, 47

Bisector, of an angle, 23
  of a line segment, 21
  perpendicular, 21

Center of gravity, 153
Center square, 239
Centroid, 153
Circle, 13
  arc of, 14
  area of, 429
  center of, 13
  central angle of, 14
  chord of, 212
  circumference of, 13, 422, 423, 425
  circumscribed, 147
  diameter of, 13
  division of, 403
  as the graph of a quadratic equation, 163
  inscribed, 402
  radius of, 13
  secant of, 212
  sector of, 212, 429
  segment of, 435
Circles, common chord of, 224
  concentric, 147
  tangent, 222
Circumcenter, 147
Collinear points, 2
Commensurable line segments, 264
Compasses, 7
  measurement with, 7
Complement of an angle, 29
Concurrent lines, 145
Congruence, defined, 52
  fundamental tests of, 59
  of polygons, 87
  of triangles, 53
Consequent, 266
Construction lines, 67
Construction problems, analytic method used in solving, 243
  method to use in solving, 90
  synthetic method used in solving, 242
  use of loci in, 245
Converse, 86
  relation of, to opposite, 137
Coplanar, 5
Corollary, 59
Cosine, 312
Cosines, law of, 371
Curvilinear figure, 6

Danger angle, 234
  horizontal, 234
Decagon, 7
Demonstrative geometry, 51
Dependence illustrated geometrically, 368
Designs, 103
Diagonal scale, 298
Dihedral angles, 201
  how to measure, 202
  how to read, 201
  kinds of, 201
  plane angle of, 201
  size of, 201
Direct variation, 270
Distance formula, 376
Division of fractions, 342
Drawing to scale, 308
Dual theorems, 79

Ellipse, 167
  as the graph of a quadratic equation, 167
  mechanical construction of, 171
Equality, of line segments, 8
  of two geometric magnitudes, 52
Equiangular polygons, 7
Equilateral polygons, 7
Equilibrant, 114
Exterior angles of a triangle, 12
Extreme and mean ratio, 410
Extremes, 265

Fallacies, geometric, 96
Formulas, applying the laws of fractions to, 338
  involving quadratic terms, 174
  midpoint, 377
Fourth proportional, 284
Fractions, addition of, 334
  complex, 344
  division of, 342
  multiplication of, 339
  subtraction of, 334
Functions, 20
  of 30°, 45°, and 60°, 317
  trigonometric, 314
Fundamental assumptions, 47

Gauss, 404
General angle, 362
Generate, 1
Geometric figure, 5
  boundaries of, 6
  curvilinear, 6
  rectilinear, 6
Geometric magnitudes, 1
Geometry, 5
  demonstrative, 51
  plane, 5
Geometry, solid, 5
  of space, 5
Graphical representation, 157
  terms used in, 157

Harmonic division, 281
Hexagon, 7
Homologous parts, 61
Hyperbola, 171
  as the graph of a quadratic equation, 171

Identities, trigonometric, 348
Incenter, 149
Inclination, 381
  of a line to a plane, 210
Incommensurable line segments, 264
Independent variable, 271
Indirect measurement, 308
Inductive method of proof, 99
Inequalities, 128
  axioms used in combining, 129
  fundamental relations of, 130
Inequality, 9, 128
Interior angles of a triangle, 12
Internal and external division of a line segment, 277
Inverse variation, 271

Kite, 75

Leibniz, 54
Length, 2
Line segment, 2
  bisection of, 21
  broken, 2
  concurrent, 145
  directed, 156
  division of, in given ratio, 281
  harmonic division of, 281

Line segment, internal and external division of, 277
  parallel to a plane, 185
Line segments, equal, 8
  properties of, 3
  unequal, 8
Lunar eclipse, 252

Mean proportional, 261
Means, 265
Measurement, of angles, 14
  of areas, 354
  indirect, 308
Median, 22
Method of proving a proposition, 62, 70
  analytic, 92
  by congruent triangles, 62
  direct, 181
  by elimination, 136
  indirect, 181
  inductive, 99
  synthetic, 92
Midpoint formulas, 377
Multiplication of fractions, 339

Napier, 326
Notation, 27
  for angles, 27
Number of solutions, 249

Obtuse angle, 11
  trigonometric functions of an, 363
Octagon, 7
Opposite, of a proposition, 136
  relation of, to converse, 137
Opposite angles, 30
Optical illusions, 49
Origin, 2
Orthocenter, 151
Outline, of angle pairs, 33
  of methods of proof, 117

Pantograph, 299
Parabola, 159
Parallel lines, 23
  how to draw, 24
Parallelogram, 25
  area of, 357
  of forces, 114
  how constructed, 26
  how used, 105
  properties of, 110
Parallel planes, 185
Pascal, 176
Pentagon, 7
Perigon, 10
Perpendicular, 20
  how to drop, 22
  how to erect, 20
  to a plane, 188
Perpendicular planes, 202
Perpendicular bisector, 21
Pi, value of, 427
Plane, 3
  how to determine, 177
Plumb level, 69
Polygon, 6
Polygons, circumscribed, 402
  convex, 104
  equal, 355
  inscribed, 147
  kinds of, 7
  reëntrant, 104
  regular, 7
Postulate, 3
Postulates, 3, 4, 13, 25, 55, 56, 224
  of motion, 54, 55, 56
Principle of Continuity illustrated, 228
Principle of Duality, 79

Prism, 119
  bases of, 119
  lateral faces of, 119
  lateral surface of, 119
  total surface of, 120
Prismatic space, 119
Prismatic surface, 119
Projection, of a point on a plane, 208
  of a line on a plane, 208
Proofs of constructions, 91
Proportion, 264
  addition law of, 336
  continued, 264
  subtraction law of, 337
  and variation, 270
Proposition, 59
Protractor, 14
Pythagorean theorem, various methods of proving, 366

Quadrant, 14
Quadratic equations, solution by formula, 160
Quadrilaterals, 7, 42
  classification of, 105, 106

Radian, 431
  value of, 432
Radian measure, 430
Radius, 13
Ratio, of similitude, 287
  of two line segments, 262
Rationalizing the denominator of a fraction, 318
Ray, 2
Rectangle, 26
  area of, 355, 356
Rectilinear figure, 6
Reductio ad absurdum proof, 182
Reflex angle, 11
Regular polygon, 7
  apothem of, 402
  center of, 402
  circumscribed, 402
  inscribed, 147, 401
  perimeter of circumscribed, 424
  perimeter of inscribed, 424
Resultant of two forces, 114
Rhombus, 26
Right angle, 10
Right triangle, 39

Scale drawings, 308
  limitations of, 309
Sector, 212, 429
Segment, 435
Semicircle, 14
Similar figures, 286
  uses of, 287
Similar polygons, 287
Similar right triangles, 310
Similar triangles, 288, 291
Simultaneous equations, 165
  quadratics, 169
Sines, law of, 383
Skew lines, 180
Skew polygons, 180
Slope, 381
Solid, 4
Solving right triangles, 313, 320
Square, 26
  area of, 356
Squared paper, 8
Straight angle, 10
Subtraction of unit fractions, 333
Superposition, 52
Symmetric figures, 57
  with respect to a center or axis, 56

Symmetric points, 56
  with respect to a straight line, 56
Symmetry, 56
Synthetic method of proof, 92

Table, of roots and powers, 439
  of sines, 312
  of trigonometric ratios, 315
Tangents, 212
  to circles, 222
Theorem, 31
Trapezoid, 43
  area of, 358
Trapezoidal rule, 361
Triangles, altitude of, 23
  angles of, 12
  base angles of, 39
  classes of, 41
  how to construct, 43
  isosceles, 39
Triangles, right, 39
Triangulation, 383
Trigonometric equations, 352
Trigonometric identities, 348
Trigonometric ratios, 313
  as functions, 314
  as functions of obtuse angles, 363
  fundamental relations of, 348
  use of, in constructing acute angles, 314

Variables, dependent, 271
  independent, 271
Variation, direct, 270
  inverse, 271
Vector, 116
Vertex of an angle, 10
Vertical angles, 30
Vieta, 352

This book should be returned to the Library on or before the last date stamped below.

A fine of five cents a day is incurred by retaining it beyond the specified time.

Please return promptly.

AUG 17 1930

AUG 16 1930

~~DUE MAY 22 '35~~

DUE JAN 30 '39

DUE JUL 29 '40